Online Appendix for "Sales-Based Compensation and Collusion with Heterogeneous Firms"

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Contents

A Price Competition

In this section, we present results under differentiated product Bertrand competition.

A.1 Preliminaries

The following assumption (assumed in the main text) ensures the industry critical discount factor is less than 1 (see Lemma A.8 for a proof). If this assumption does not hold, collusion is unsustainable for any discount factor. We assume the following for the remainder of Section A. Recall that x is defined as the asymmetry in perceived marginal cost, $x := \tilde{c}_2 - \tilde{c}_1$.

Assumption 1. *$x - a \in (y_L, y_U)$ where $y_L = (1-\tilde{c}_1)\left(1 - \frac{(4-b^2)\sqrt{b^2+8}-b^3}{8}\right)$ and $y_U = (1-\tilde{c}_1)\left(1 - \frac{(4-b^2)\sqrt{b^2+8}+b^3}{16-6b^2}\right)$.*

The following lemma will be used throughout Section A.

Lemma A.1. *i) $y_L < x-a < y_U \implies -(1-\tilde{c}_1)(1-b) < x-a < (1-\tilde{c}_1)(1-b)$,*
ii) $y_L < x-a < y_U \implies (1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right) < x-a < (1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right)$, and
iii) $y_L < x-a < y_U \implies (1-\tilde{c}_1)\left(1-\frac{1}{b}\right) < x-a < (1-\tilde{c}_1)(1-b)$.

Proof. Part i) The proof follows from[1]

$$
\begin{aligned}
& y_U = (1-\tilde{c}_1)\left(1 - \frac{(4-b^2)\sqrt{b^2+8}+b^3}{16-6b^2}\right) < (1-\tilde{c}_1)(1-b) \\
\iff \quad & 1 - \frac{(4-b^2)\sqrt{b^2+8}+b^3}{16-6b^2} < 1-b \\
\iff \quad & -\frac{(4-b^2)\sqrt{b^2+8}+b^3}{16-6b^2} < -b \\
\iff \quad & b < \frac{(4-b^2)\sqrt{b^2+8}+b^3}{16-6b^2} \\
\iff \quad & 16b - 6b^3 < (4-b^2)\sqrt{b^2+8}+b^3 \\
\iff \quad & 16b < (4-b^2)\sqrt{b^2+8}+7b^3
\end{aligned}
$$

which holds for all $b > 0$, and

$$
\begin{aligned}
& y_L = (1-\tilde{c}_1)\left(1 - \frac{(4-b^2)\sqrt{(b^2+8)}-b^3}{8}\right) > -(1-\tilde{c}_1)(1-b) \\
\iff \quad & 1 - \frac{(4-b^2)\sqrt{(b^2+8)}-b^3}{8} > -1+b \\
\iff \quad & -\frac{(4-b^2)\sqrt{(b^2+8)}-b^3}{8} > -2+b \\
\iff \quad & 2-b > \frac{(4-b^2)\sqrt{(b^2+8)}-b^3}{8} \\
\iff \quad & 16-8b > (4-b^2)\sqrt{b^2+8}-b^3
\end{aligned}
$$

which holds for all $b > 0$.

[1] Note that $\tilde{c}_1 \le c_1 < 1$ where the last inequality follows by assumption.

Part ii) The proof follows from Part i,

$$
\begin{aligned}
& (1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right) < -(1-\tilde{c}_1)(1-b) \\
\iff \quad & 1-\frac{2-b^2}{b} < -1+b \\
\iff \quad & 2-\frac{2-b^2}{b} < b \\
\iff \quad & 2-b < \frac{2-b^2}{b} \\
\iff \quad & 2b-b^2 < 2-b^2 \\
\iff \quad & b < 1,
\end{aligned}
$$

and

$$
\begin{aligned}
& (1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right) > (1-\tilde{c}_1)(1-b) \\
\iff \quad & b > \frac{b}{2-b^2} \\
\iff \quad & 2-b^2 > 1 \\
\iff \quad & 1 > b^2.
\end{aligned}
$$

Part iii) The proof follows from Part i and

$$
\begin{aligned}
& (1-\tilde{c}_1)\left(1-\frac{1}{b}\right) < -(1-\tilde{c}_1)(1-b) \\
\iff \quad & 1-\frac{1}{b} < -(1-b) \\
\iff \quad & 1-\frac{1}{b} < -1+b \\
\iff \quad & 2-\frac{1}{b} < b \\
\iff \quad & 2b-1 < b^2 \\
\iff \quad & 0 < b^2-2b+1 \\
\iff \quad & 0 < (b-1)^2.
\end{aligned}
$$

□

Equivalence of Contract Types Let

$$M_i(p_1,p_2;\theta_i) = (1-\theta_i)\,\pi_i(p_1,p_2) + \theta_i s_i(p_1,p_2) \tag{1}$$

denote manager i's payoff when prices are p_1 and p_2 and managerial compensation is a convex combination of profit ($\pi_i(p_1,p_2)$) and sales ($s_i(p_1,p_2)$). Equation (1) represents the Fershtman and Judd (1987) specification referenced in Appendix A of Lambertini and Trombetta (2002).

Let

$$M_i^{LT}(p_1,p_2;\alpha_i) = \pi_i(p_1,p_2) + \alpha_i D_i(p_1,p_2) \tag{2}$$

denote manager i's payoff when prices are p_1 and p_2 and managerial compensation equals profit $\pi_i(p_1,p_2)$ plus the quantity sold times a positive weight α_i. This specification was employed in Lambertini and Trombetta (2002). We show that these contracts are equivalent when $\alpha_i = \theta_i c_i$. In other words $M_i^{LT}(p_1,p_2;\theta_i c_i) = M_i(p_1,p_2;\theta_i)$.

Simplifying (1) and using $\alpha_i = \theta_i c_i$ yields

$$\begin{aligned} M_i(p_1, p_2; \theta_i) &= (1-\theta_i)\,\pi_i + \theta_i s_i \\ &= (1-\theta_i)\, D_i(p_1,p_2)(p_i - c_i) + \theta_i D_i(p_1,p_2) p_i \\ &= D_i(p_1,p_2)(p_i - c_i) - \theta_i D_i(p_1,p_2)(p_i - c_i) + \theta_i D_i(p_1,p_2) p_i \\ &= D_i(p_1,p_2)(p_i - c_i) - \theta_i D_i(p_1,p_2)(-c_i) \\ &= D_i(p_1,p_2)(p_i - c_i) + \theta_i D_i(p_1,p_2) c_i \\ &= D_i(p_1,p_2)(p_i - c_i) + \alpha_i D_i(p_1,p_2) \\ &= M_i^{LT}(p_1, p_2; \theta_i c_i)\,. \end{aligned}$$

Additional Technical Lemmas The following technical lemmas will be used to derive critical discount factors, and prove Lemma 1 and Lemma 2 from the text appendix.

Lemma A.2. $A_2(\tilde{c}_1, x, a) = \frac{b^2(1-\tilde{c}_1)^2}{((2-b)(1-\tilde{c}_1)-2(x-a))^2 - \frac{16}{(4-b^2)^2}((2-b-b^2)(1-\tilde{c}_1)-(2-b^2)(x-a))^2}$ *is increasing in* $x-a$ *if* $\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)} > \frac{2b}{2-b^2}$.

Proof. Note that $A_2(\tilde{c}_1, x, a) = A_2(\tilde{c}_1, y, 0)$ where y is defined as $y = x - a$. Also, note that

$$\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)} = \frac{1-\tilde{c}_1}{1-(\tilde{c}_1+y)} > \frac{2b}{2-b^2} \iff y > y_2^* = (1-\tilde{c}_1)\left(1 - \frac{2-b^2}{2b}\right).$$

The remainder of the proof shows that $\frac{\partial}{\partial y}A_2(\tilde{c}_1, y, 0) > 0$ for y such that $y \in (\max\{y_L, y_2^*\}, y_U)$. $\frac{\partial}{\partial y}A_2(\tilde{c}_1, y, 0) > 0$ when

$$\begin{aligned} -b^2(1-\tilde{c}_1)^2\left(-4(-b-2(\tilde{c}_1+y)+b\tilde{c}_1+2) + (2-b^2)\frac{32}{(4-b^2)^2}\left(-b-2(\tilde{c}_1+y)+b\tilde{c}_1+b^2(\tilde{c}_1+y)-b^2+2\right)\right) &> 0 \\ -4(-b-2(\tilde{c}_1+y)+b\tilde{c}_1+2) + (2-b^2)\frac{32}{(4-b^2)^2}\left(-b-2(\tilde{c}_1+y)+b\tilde{c}_1+b^2(\tilde{c}_1+y)-b^2+2\right) &< 0 \\ 8y - 4(-b-2\tilde{c}_1+b\tilde{c}_1+2) + (2-b^2)\frac{32}{(4-b^2)^2}\left(-b-2\tilde{c}_1+b\tilde{c}_1+b^2\tilde{c}_1-b^2+2\right) - (2-b^2)^2\frac{32}{(4-b^2)^2}y &< 0 \\ 8y - 4(2-b)(1-\tilde{c}_1) + (2-b^2)\frac{32}{(4-b^2)^2}(1-\tilde{c}_1)(2-b-b^2) - (2-b^2)^2\frac{32}{(4-b^2)^2}y &< 0 \\ y\left(8 - (2-b^2)^2\frac{32}{(4-b^2)^2}\right) - \left(4(2-b)(1-\tilde{c}_1) - (2-b^2)\frac{32}{(4-b^2)^2}(1-\tilde{c}_1)(2-b-b^2)\right) &< 0 \\ y\left(4 - (2-b^2)^2\frac{16}{(4-b^2)^2}\right) + \left((2-b^2)\frac{16}{(4-b^2)^2}(1-\tilde{c}_1)(2-b-b^2) - 2(2-b)(1-\tilde{c}_1)\right) &< 0. \end{aligned} \tag{3}$$

Thus, $\frac{\partial}{\partial y}A_2(\tilde{c}_1, y, 0) > 0$ if

$$F_1(y) = \Bigg(\underbrace{4 - (2-b^2)^2\frac{16}{(4-b^2)^2}}_{\text{Term 1}}\Bigg)y - \Bigg(\underbrace{(2-b^2)\frac{16}{(4-b^2)^2}(1-\tilde{c}_1)(2-b-b^2) - 2(2-b)(1-\tilde{c}_1)}_{\text{Term 2}}\Bigg) < 0.$$

Note that Term 1 is positive by

$$\begin{aligned}
4-\left(2-b^2\right)^2\frac{16}{\left(4-b^2\right)^2} &> 0\\
\left(4-b^2\right)^2 &> 4\left(2-b^2\right)^2\\
4-b^2 &> 2\left(2-b^2\right)\\
4-b^2 &> 4-2b^2\\
b^2 &> 0
\end{aligned}$$

and Term 2 is negative by

$$\begin{aligned}
\frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)(1-\tilde{c}_1)\left(2-b-b^2\right)-2\left(2-b\right)(1-\tilde{c}_1) &< 0\\
\frac{8}{\left(4-b^2\right)^2}\left(2-b^2\right)\left(2-b-b^2\right)-(2-b) &< 0\\
8\left(2-b^2\right)\left(2-b-b^2\right) &< (2-b)\left(4-b^2\right)^2\\
8\left(4-2b-2b^2-2b^2-b^3+b^4\right) &< (2-b)\left(16-8b^2+b^4\right)\\
32-16b-16b^2-16b^2-8b^3+8b^4 &< 32-16b^2+2b^4-16b+8b^3-b^5\\
-16b^2-8b^3+8b^4 &< 2b^4+8b^3-b^5\\
b^5+6b^4 &< 16b^2+16b^3\\
b^3+6b^2 &< 16+16b.
\end{aligned}$$

Thus, $F_1(y)$ is increasing in y. Therefore, it is sufficient to show that $F_1((1-b)(1-\tilde{c}_1)) < 0$ because $y_U < (1-b)(1-\tilde{c}_1)$ (where the inequality follows from Lemma A.1) and $F_1(y)$ is increasing in y.

$$
\begin{aligned}
F_1((1-b)(1-\tilde{c}_1)) &< 0 \\
(1-\tilde{c}_1)(1-b)\left(4-\frac{16}{(4-b^2)^2}\left(2-b^2\right)^2\right)+(1-\tilde{c}_1)\left(\frac{16}{(4-b^2)^2}\left(2-b^2\right)\left(2-b-b^2\right)-2(2-b)\right) &< 0 \\
(1-b)\left(4-\frac{16}{(4-b^2)^2}\left(2-b^2\right)^2\right)+\left(\frac{16}{(4-b^2)^2}\left(2-b^2\right)\left(2-b-b^2\right)-2(2-b)\right) &< 0 \\
\left(4-\frac{16}{(4-b^2)^2}\left(2-b^2\right)^2+\frac{16}{(4-b^2)^2}\left(2-b^2\right)\left(2-b-b^2\right)-2(2-b)\right)-b\left(4-\frac{16}{(4-b^2)^2}\left(2-b^2\right)^2\right) &< 0 \\
\left(4-b\frac{16}{(4-b^2)^2}\left(2-b^2\right)-2(2-b)\right)-b\left(4-\frac{16}{(4-b^2)^2}\left(2-b^2\right)^2\right) &< 0 \\
\left(-b\frac{16}{(4-b^2)^2}\left(2-b^2\right)+2b\right)-b\left(4-\frac{16}{(4-b^2)^2}\left(2-b^2\right)^2\right) &< 0 \\
-2b-b\frac{16}{(4-b^2)^2}\left(2-b^2\right)+\frac{16}{(4-b^2)^2}\left(2-b^2\right)^2 b &< 0 \\
-1-\frac{8}{(4-b^2)^2}\left(2-b^2\right)+\frac{8}{(4-b^2)^2}\left(2-b^2\right)^2 &< 0 \\
-1+\frac{8}{(4-b^2)^2}\left(2-b^2\right)\left(1-b^2\right) &< 0 \\
8\left(2-b^2\right)\left(1-b^2\right) &< \left(4-b^2\right)^2 \\
8\left(2-2b^2-b^2+b^4\right) &< 16-8b^2+b^4 \\
8\left(2-2b^2-b^2+b^4\right) &< 16-8b^2+b^4 \\
16-16b^2-8b^2+8b^4 &< 16-8b^2+b^4 \\
-16b^2+7b^4 &< 0 \\
7b^2 &< 16
\end{aligned}
$$

which holds as $b < 1$. □

Lemma A.3. $B_2(\tilde{c}_1, x, a) = \frac{\left(\frac{1-\tilde{c}_1}{2b}\right)\left(\frac{2b+2ab+\tilde{c}_1-1}{2b}-\tilde{c}_2\right)-\frac{1}{1-b^2}\left(\frac{1-\tilde{c}_2+a}{2}\right)\left(\frac{1-\tilde{c}_2+a-b(1-\tilde{c}_1)}{2}\right)}{\left(\frac{1-\tilde{c}_1}{2b}\right)\left(\frac{2b+2ab+\tilde{c}_1-1}{2b}-\tilde{c}_2\right)-\frac{\left((2-b^2)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1)\right)^2}{(4-b^2)^2(1-b^2)}}$ *is increasing in* $x - a$ *if* $\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)} < \frac{2b}{2-b^2}$.

Proof. Note that $B_2(\tilde{c}_1, x, a) = B_2(\tilde{c}_1, y, 0)$ where y is defined as $y := x - a$. Also, note that

$$\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)} = \frac{1-\tilde{c}_1}{1-(\tilde{c}_1+y)} < \frac{2b}{2-b^2} \iff y < y_2^* = (1-\tilde{c}_1)\left(1-\frac{2-b^2}{2b}\right).$$

The remainder of the proof shows that $\frac{\partial}{\partial y}B_2(\tilde{c}_1, y, 0) > 0$ for y such that $y \in (y_L, \min\{y_2^*, y_U\})$. Routine computations show that $\frac{\partial}{\partial y}B_2(\tilde{c}_1, y, 0) > 0$ when

$$M_1 y^2 + M_2 y + M_3 > 0$$

where

$$
\begin{aligned}
M_1 &= 2b^5 - 14b^3 + 16b \\
M_2 &= 32b\tilde{c}_1 - 16\tilde{c}_1 - 32b + 14b^2\tilde{c}_1 - 28b^3\tilde{c}_1 \\
&\quad - 6b^4\tilde{c}_1 + 4b^5\tilde{c}_1 - 14b^2 + 28b^3 + 6b^4 - 4b^5 + 16 \\
M_3 &= b^5\tilde{c}_1^2 - 2b^5\tilde{c}_1 + b^5 - 6b^4\tilde{c}_1^2 + 12b^4\tilde{c}_1 - 6b^4 - 17b^3\tilde{c}_1^2 + 34b^3\tilde{c}_1 - 17b^3 \\
&\quad + 14b^2\tilde{c}_1^2 - 28b^2\tilde{c}_1 + 14b^2 + 24b\tilde{c}_1^2 - 48b\tilde{c}_1 + 24b - 16\tilde{c}_1^2 + 32\tilde{c}_1 - 16.
\end{aligned}
$$

$M_1y^2 + M_2y + M_3$ is an upward facing parabola with roots (by the quadratic formula) given by

$$
\begin{aligned}
r &= \frac{(16b + 8\tilde{c}_1 \pm D - 16b\tilde{c}_1 - 7b^2\tilde{c}_1 + 14b^3\tilde{c}_1 + 3b^4\tilde{c}_1 - 2b^5\tilde{c}_1 + 7b^2 - 14b^3 - 3b^4 + 2b^5 - (\pm\tilde{c}_1 D) - 8)}{2(b^5 - 7b^3 + 8b)} \\
&= \frac{(16b\,(1-\tilde{c}_1) - 8\,(1-\tilde{c}_1) \pm D\,(1-\tilde{c}_1) + 7b^2\,(1-\tilde{c}_1) - 14b^3\,(1-\tilde{c}_1) - 3b^4\,(1-\tilde{c}_1) + 2b^5\,(1-\tilde{c}_1))}{2(b^5 - 7b^3 + 8b)} \\
&= (1-\tilde{c}_1)\,\frac{(16b - 8 \pm D + 7b^2 - 14b^3 - 3b^4 + 2b^5)}{2(b^5 - 7b^3 + 8b)} \\
&= (1-\tilde{c}_1)\left(1 + \frac{-8 \pm D + 7b^2 - 3b^4}{2(b^5 - 7b^3 + 8b)}\right)
\end{aligned}
$$

where $D = \sqrt{(b^2+8)(b-1)(b+1)(2b^6 - 13b^4 + 23b^2 - 8)}$. If the roots are complex, $M_1y^2 + M_2y + M_3 > 0$ always holds and the proof is complete. If one or more roots are real, then the proof is complete if the smaller of the two roots is greater than $\min\{y_2^*, y_U\}$. Let $r_L = (1-\tilde{c}_1)\left(1 + \frac{-8-D+7b^2-3b^4}{2(b^5-7b^3+8b)}\right)$ denote the smaller of the two roots.

$$
\begin{aligned}
y_2^* &< r_L \\
y_2^* = (1-\tilde{c}_1)\left(1 - \frac{2-b^2}{2b}\right) &< (1-\tilde{c}_1)\left(1 + \frac{-8 - D + 7b^2 - 3b^4}{2(b^5 - 7b^3 + 8b)}\right) \\
1 - \frac{2-b^2}{2b} &< 1 + \frac{-8 - D + 7b^2 - 3b^4}{2(b^5 - 7b^3 + 8b)} \\
-\frac{2-b^2}{2b} &< \frac{-8 - D + 7b^2 - 3b^4}{2\,(b^5 - 7b^3 + 8b)}
\end{aligned}
$$

which holds. □

Lemma A.4. $A_1(\tilde{c}_1, x, a) = \frac{b^2(1-(\tilde{c}_1+x)+a)^2}{(b+2\tilde{c}_1+ab-b(\tilde{c}_1+x)-2)^2 - \frac{16\left(b+2\tilde{c}_1+ab-b(\tilde{c}_1+x)-b^2\tilde{c}_1+b^2-2\right)^2}{(4-b^2)^2}}$ *is decreasing in* $x-a$ *if* $\frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} > \frac{2b}{2-b^2}$.

Proof. Note that $A_1(\tilde{c}_1, x, a) = A_1(\tilde{c}_1, y, 0)$ where y is defined as $y = x - a$. Also, note that

$$
\frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} = \frac{1-(\tilde{c}_1+y)}{1-\tilde{c}_1} > \frac{2b}{2-b^2} \iff y < y_1^* = (1-\tilde{c}_1)\left(1 - \frac{2b}{2-b^2}\right).
$$

Thus, the remainder of the proof, for this case, shows that $\frac{\partial}{\partial y}A_1(\tilde{c}_1, y, 0) < 0$ for y such that $y \in (y_L, \min\{y_U, y_1^*\})$. $\frac{\partial}{\partial y}A_1(\tilde{c}_1, y, 0) < 0$ when

$$
\begin{aligned}
&-\left(((2-b)\,(1-\tilde{c}_1) + by)^2 - \frac{16\left(\left(2-b-b^2\right)(1-\tilde{c}_1) + by\right)^2}{(4-b^2)^2}\right) 2b^2\,(1-\tilde{c}_1-y) \\
&-b^2\,(1-\tilde{c}_1-y)^2\left(2b\,((2-b)\,(1-\tilde{c}_1) + by) - \frac{32b\left(\left(2-b-b^2\right)(1-\tilde{c}_1) + by\right)}{(4-b^2)^2}\right) < 0
\end{aligned}
$$

Dividing both sides by $-2b^2(1-\tilde{c}_1-y)$ (note that this quantity is negative) and simplifying yields

$$\begin{aligned}
&\left(\left(\left(2-b\right)\left(1-\tilde{c}_1\right)+by\right)^2-\frac{16\left(\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)+by\right)^2}{\left(4-b^2\right)^2}\right)\\
&+\left(1-\tilde{c}_1-y\right)\left(b\left(\left(2-b\right)\left(1-\tilde{c}_1\right)+by\right)-\frac{16b\left(\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)+by\right)}{\left(4-b^2\right)^2}\right)>0\\
&\left(\left(2-b\right)\left(1-\tilde{c}_1\right)\right)^2+2by\left(2-b\right)\left(1-\tilde{c}_1\right)+b^2y^2\\
&-\frac{16}{\left(4-b^2\right)^2}\left(\left(\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)\right)^2+2by\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)+b^2y^2\right)\\
&+\left(1-\tilde{c}_1-y\right)\left(b\left(\left(2-b\right)\left(1-\tilde{c}_1\right)+by\right)-\frac{16b\left(\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)+by\right)}{\left(4-b^2\right)^2}\right)>0\\
&\left(\left(2-b\right)\left(1-\tilde{c}_1\right)\right)^2+2by\left(2-b\right)\left(1-\tilde{c}_1\right)+b^2y^2\\
&-\frac{16}{\left(4-b^2\right)^2}\left(\left(\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)\right)^2+2by\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)+b^2y^2\right)\\
&+\left(1-\tilde{c}_1\right)\left(b\left(\left(2-b\right)\left(1-\tilde{c}_1\right)+by\right)-\frac{16b\left(\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)+by\right)}{\left(4-b^2\right)^2}\right)\\
&-y\left(b\left(\left(2-b\right)\left(1-\tilde{c}_1\right)+by\right)-\frac{16b\left(\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)+by\right)}{\left(4-b^2\right)^2}\right)>0.
\end{aligned}$$

Continuing to simplify yields

$$
\begin{aligned}
&((2-b)(1-\tilde{c}_1))^2 + 2by(2-b)(1-\tilde{c}_1) + b^2y^2 \\
&-\frac{16}{(4-b^2)^2}\left(\left(2-b-b^2\right)(1-\tilde{c}_1)\right)^2 \\
&-\frac{16}{(4-b^2)^2}2by\left(2-b-b^2\right)(1-\tilde{c}_1) - \frac{16}{(4-b^2)^2}b^2y^2 \\
&+\left(b(2-b)(1-\tilde{c}_1)^2 + b^2y(1-\tilde{c}_1) - \frac{16b\left(2-b-b^2\right)(1-\tilde{c}_1)^2 + 16b^2y(1-\tilde{c}_1)}{(4-b^2)^2}\right) \\
&-\left(b(2-b)(1-\tilde{c}_1)y + b^2y^2 - \frac{16b\left(2-b-b^2\right)(1-\tilde{c}_1)y + 16b^2y^2}{(4-b^2)^2}\right) > 0
\end{aligned}
$$

$$
\begin{aligned}
&((2-b)(1-\tilde{c}_1))^2 + by(2-b)(1-\tilde{c}_1) \\
&-\frac{16}{(4-b^2)^2}\left(\left(2-b-b^2\right)(1-\tilde{c}_1)\right)^2 - \frac{16}{(4-b^2)^2}by\left(2-b-b^2\right)(1-\tilde{c}_1) \\
&+\left(b(2-b)(1-\tilde{c}_1)^2 + b^2y(1-\tilde{c}_1) - \frac{16b\left(2-b-b^2\right)(1-\tilde{c}_1)^2 + 16b^2y(1-\tilde{c}_1)}{(4-b^2)^2}\right) > 0
\end{aligned}
$$

$$
\begin{aligned}
&2(2-b)(1-\tilde{c}_1)^2 + 2by(1-\tilde{c}_1) \\
&-\frac{16}{(4-b^2)^2}\left(\left(2-b-b^2\right)(1-\tilde{c}_1)\right)^2 - \frac{16}{(4-b^2)^2}by\left(2-b-b^2\right)(1-\tilde{c}_1) \\
&-\frac{16b\left(2-b-b^2\right)(1-\tilde{c}_1)^2 + 16b^2y(1-\tilde{c}_1)}{(4-b^2)^2} > 0
\end{aligned}
$$

$$
\begin{aligned}
&2(2-b)(1-\tilde{c}_1)^2 + 2by(1-\tilde{c}_1) \\
&-\frac{16}{(4-b^2)^2}\left(2-b-b^2\right)\left(2-b^2\right)((1-\tilde{c}_1))^2 - \frac{16}{(4-b^2)^2}by\left(2-b^2\right)(1-\tilde{c}_1) > 0
\end{aligned}
$$

$$
2(2-b)(1-\tilde{c}_1) + 2by - \frac{16}{(4-b^2)^2}\left(2-b-b^2\right)\left(2-b^2\right)(1-\tilde{c}_1) - \frac{16}{(4-b^2)^2}by\left(2-b^2\right) > 0
$$

$$
y\left(2b - \frac{16}{(4-b^2)^2}b\left(2-b^2\right)\right) + 2(2-b)(1-\tilde{c}_1) - \frac{16}{(4-b^2)^2}\left(2-b-b^2\right)\left(2-b^2\right)(1-\tilde{c}_1) > 0
$$

$$
y\left(2b - \frac{16}{(4-b^2)^2}b\left(2-b^2\right)\right) + (1-\tilde{c}_1)\left(2(2-b) - \frac{16}{(4-b^2)^2}\left(2-b-b^2\right)\left(2-b^2\right)\right) > 0
$$

Thus, $\frac{\partial}{\partial y}A_1(\tilde{c}_1, y, 0) < 0$ if

$$
F_2(y) = y\left(\underbrace{2b - \frac{16}{(4-b^2)^2}\left(2-b^2\right)b}_{\text{Term 1}}\right) + (1-\tilde{c}_1)\left(\underbrace{2(2-b) - \frac{16}{(4-b^2)^2}\left(2-b^2\right)\left(2-b-b^2\right)}_{\text{Term 2}}\right) > 0.
$$

Note that Term 1 is positive by

$$\begin{aligned}
2b - \frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)b &> 0 \\
2b &> \frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)b \\
1 &> \frac{8}{\left(4-b^2\right)^2}\left(2-b^2\right) \\
\left(4-b^2\right)^2 &> 8\left(2-b^2\right) \\
16-8b^2+b^4 &> 16-8b^2 \\
b^4 &> 0.
\end{aligned}$$

Note that Term 2 is positive by

$$\begin{aligned}
2\left(2-b\right) - \frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)\left(2-b-b^2\right) &> 0 \\
2(2-b)\left(4-b^2\right)^2 &> 16\left(2-b^2\right)\left(2-b-b^2\right) \\
(2-b)\left(4-b^2\right)^2 &> 8\left(2-b^2\right)\left(2-b-b^2\right) \\
(2-b)\left(16-8b^2+b^4\right) &> 8\left(4-2b-2b^2-2b^2+b^3+b^4\right) \\
(2-b)\left(16-8b^2+b^4\right) &> 8\left(4-2b-4b^2+b^3+b^4\right) \\
32-16b^2+2b^4-\left(16b-8b^3+b^5\right) &> 32-16b-32b^2+8b^3+8b^4 \\
32-16b^2+2b^4-16b+8b^3-b^5 &> 32-16b-32b^2+8b^3+8b^4 \\
16b^2-b^5 &> 6b^4 \\
16b^2 &> 6b^4+b^5 \\
16 &> b^2+b^3.
\end{aligned}$$

Thus, $F_2(y)$ is increasing in y. Therefore, it is sufficient to show that $F_2(-(1-\tilde{c}_1)(1-b)) > 0$ because

$-(1-\tilde{c}_1)(1-b) < y_L$ and $F_2(y)$ is increasing in y.

$$
\begin{aligned}
F_2(y) &> 0 \\
-(1-\tilde{c}_1)(1-b)\left(2b - \frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)b\right) & \\
+\left(1-\tilde{c}_1\right)\left(2\left(2-b\right) - \frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)\left(2-b-b^2\right)\right) &> 0 \\
-(1-b)\left(2b - \frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)b\right) & \\
+\left(2\left(2-b\right) - \frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)\left(2-b-b^2\right)\right) &> 0 \\
-\left(2b - 2b^2 - \frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)b + b^2\frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)\right) & \\
+\left(4 - 2b - \frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)\left(2-b-b^2\right)\right) &> 0 \\
-2b + 2b^2 + \frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)b - b^2\frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right) & \\
+4 - 2b - \frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)\left(2-b-b^2\right) &> 0 \\
4 - 2b + 2b^2 + \frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)\left(b - b^2 - \left(2-b-b^2\right)\right) &> 0 \\
4 - 4b + 2b^2 + \frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)\left(2b-2\right) &> 0 \\
2 - 2b + b^2 - \frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)\left(1-b\right) &> 0 \\
2 - 2b + b^2 &> \frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)\left(1-b\right) \\
\left(4-b^2\right)^2\left(2-2b+b^2\right) &> 16\left(2-b^2\right)\left(1-b\right) \\
b^4 + 16 &> 2b^2(b+3)
\end{aligned}
$$

which holds for all b. $\square$

Lemma A.5. $B_1(\tilde{c}_1, x, a) = \frac{\frac{1-\tilde{c}_2+a}{2b}\left(\frac{2b-1+\tilde{c}_2-a}{2b}-\tilde{c}_1\right)-\frac{1}{1-b^2}\left(\frac{1-\tilde{c}_1}{2}\right)\left(\frac{1-\tilde{c}_1-b(1-\tilde{c}_2+a)}{2}\right)}{\frac{1-\tilde{c}_2+a}{2b}\left(\frac{2b-1+\tilde{c}_2-a}{2b}-\tilde{c}_1\right)-\frac{\left((2-b^2)(1-\tilde{c}_1)-b(1-\tilde{c}_2+a)\right)^2}{(4-b^2)^2(1-b^2)}}$ *is decreasing in* $x-a$ *if* $\frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} < \frac{2b}{2-b^2}$.

Proof. Note that $B_1(\tilde{c}_1, x, a) = B_1(\tilde{c}_1, y, 0)$ where y is defined as $y := x - a$. Also, note that

$$
\frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} = \frac{1-(\tilde{c}_1+y)}{1-\tilde{c}_1} < \frac{2b}{2-b^2} \iff y > y_1^* = (1-\tilde{c}_1)\left(1 - \frac{2b}{2-b^2}\right).
$$

The remainder of the proof shows that $\frac{\partial}{\partial y}B_1(\tilde{c}_1, y, 0) < 0$ for y such that $y \in (\max\{y_L, y_1^*\}, y_U)$. Routine computations show that $\frac{\partial}{\partial y}B_1(\tilde{c}_1, y, 0) < 0$ when

$$
M_1 y^2 + M_2 y + M_3 > 0
$$

where

$$
\begin{aligned}
M_1 &= -b^5 - 3b^3 + 8b \\
M_2 &= 16b\tilde{c}_1 - 16\tilde{c}_1 - 16b + 14b^2\tilde{c}_1 - 6b^3\tilde{c}_1 - 6b^4\tilde{c}_1 \\
&\quad - 2b^5\tilde{c}_1 - 14b^2 + 6b^3 + 6b^4 + 2b^5 + 16 \\
M_3 &= b^5\tilde{c}_1^2 - 2b^5\tilde{c}_1 + b^5 - 6b^4\tilde{c}_1^2 + 12b^4\tilde{c}_1 - 6b^4 - 17b^3\tilde{c}_1^2 + 34b^3\tilde{c}_1 \\
&\quad - 17b^3 + 14b^2\tilde{c}_1^2 - 28b^2\tilde{c}_1 + 14b^2 + 24b\tilde{c}_1^2 - 48b\tilde{c}_1 + 24b - 16\tilde{c}_1^2 + 32\tilde{c}_1 - 16.
\end{aligned}
$$

$M_1y^2 + M_2y + M_3$ is an upward facing parabola with roots (by the quadratic formula) given by

$$
\begin{aligned}
r &= \frac{8b + 8\tilde{c}_1 \pm D - 8b\tilde{c}_1 - 7b^2\tilde{c}_1 + 3b^3\tilde{c}_1 + 3b^4\tilde{c}_1 + b^5\tilde{c}_1 + 7b^2 - 3b^3 - 3b^4 - b^5 - (\pm\tilde{c}_1 D) - 8}{8b - b^5 - 3b^3} \\
&= \frac{8b\left(1-\tilde{c}_1\right) + 8\left(\tilde{c}_1 - 1\right) \pm D\left(1-\tilde{c}_1\right) - 3b^3\left(1-\tilde{c}_1\right) - 3b^4\left(1-\tilde{c}_1\right) + b^5\left(\tilde{c}_1 - 1\right) + 7b^2\left(1-\tilde{c}_1\right)}{8b - b^5 - 3b^3} \\
&= \left(1-\tilde{c}_1\right)\frac{8b - 8 \pm D - 3b^3 - 3b^4 - b^5 + 7b^2}{8b - b^5 - 3b^3} \\
&= \left(1-\tilde{c}_1\right)\left(1 + \frac{-8 \pm D - 3b^4 + 7b^2}{8b - 3b^3 - b^5}\right) \\
&= \left(1-\tilde{c}_1\right)\left(1 - \frac{8 \pm D + 3b^4 - 7b^2}{8b - 3b^3 - b^5}\right)
\end{aligned}
$$

where $D = \sqrt{(b^2+8)(b-1)(b+1)(2b^6 - 13b^4 + 23b^2 - 8)}$. If the roots are complex, $M_1y^2 + M_2y + M_3 > 0$ always holds and the proof is complete. If one or more roots are real, then the proof is complete if the larger of the two roots is less than y_1^*. Let $r_U = \left(1-\tilde{c}_1\right)\left(1 - \frac{8-D+3b^4-7b^2}{8b-3b^3-b^5}\right)$ denote the larger of the two roots.

$$
\begin{aligned}
r_U &< y_1^* \\
\left(1-\tilde{c}_1\right)\left(1 - \frac{8 - D + 3b^4 - 7b^2}{8b - 3b^3 - b^5}\right) &< \left(1-\tilde{c}_1\right)\left(1 - \frac{2b}{2-b^2}\right) \\
1 - \frac{8 - D + 3b^4 - 7b^2}{8b - 3b^3 - b^5} &< 1 - \frac{2b}{2-b^2} \\
\frac{2b}{2-b^2} &< \frac{8 - D + 3b^4 - 7b^2}{8b - 3b^3 - b^5}
\end{aligned}
$$

which holds. □

A.2 Payoffs

In this subsection, we derive manager payoffs in each phase.

A.2.1 Competitive Phase

The analysis in this subsection is based on Zanchettin (2006) (specifically, the analysis of the case where both firms are active in equilibrium). In the competitive phase, manager 1 solves

$$
\begin{aligned}
&\max_{p_1} M_1(p_1, p_2) \\
&\max_{p_1} \left(1-\theta_1\right) D_1(p_1, p_2)(p_1 - c_1) + \theta_1 D_1(p_1, p_2) p_1 \\
&\max_{p_1} D_1(p_1, p_2)\left(p_1 - \left(1-\theta_1\right) c_1\right) \\
&\max_{p_1} D_1(p_1, p_2)(p_1 - \tilde{c}_1) \\
&\max_{p_1} \frac{1}{1-b^2}\left[1 - b - ba - p_1 + bp_2\right](p_1 - \tilde{c}_1)
\end{aligned}
$$

which yields a best reply function of $p_1(p_2) = \frac{1+\tilde{c}_1-b-ab+bp_2}{2}$. Manager 2 solves

$$\begin{aligned}\max_{p_2} M_2(p_1,p_2)\\ \max_{p_2}(1-\theta_2)\,D_2(p_1,p_2)(p_2-c_2)+\theta_2 D_2(p_1,p_2)p_2\\ \max_{p_2} D_2(p_1,p_2)\,(p_2-(1-\theta_2)\,c_2)\\ \max_{p_2} D_2(p_1,p_2)(p_2-\tilde{c}_2)\\ \max_{p_2}\frac{1}{1-b^2}\left[1-b+a-p_2+bp_1\right](p_2-\tilde{c}_2)\end{aligned}$$

which yields a best reply function of $p_2(p_1) = \frac{1+\tilde{c}_2-b+a+bp_1}{2}$. Solving for the intersection of the best replies yields equilibrium prices:

$$p_1^N = \frac{2-b+2\tilde{c}_1-ab+b\tilde{c}_2-b^2}{4-b^2} \tag{4}$$

and

$$p_2^N = \frac{2-b+2\tilde{c}_2+2a+b\tilde{c}_1-ab^2-b^2}{4-b^2}. \tag{5}$$

Zanchettin (2006) shows that both firms are active in the Nash Equilibrium if

$$(1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right) > x-a > (1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right)$$

which holds by Assumption 1 and Lemma A.1.

Additionally, note that

$$\begin{aligned}
& D_1(p_1^N,p_2^N) = \frac{1}{1-b^2}\left[1-b-ba-p_1^N+bp_2^N\right] > 0\\
\iff\ & 1-b-ba-\frac{2-b+2\tilde{c}_1-ab+b\tilde{c}_2-b^2}{4-b^2}\\
& +b\frac{2-b+2\tilde{c}_2+2a+b\tilde{c}_1-ab^2-b^2}{4-b^2} > 0\\
\iff\ & 4-b^2-4b+b^3-4ba+b^3a-\left(2-b+2\tilde{c}_1-ab+b\tilde{c}_2-b^2\right)\\
& +\left(2b-b^2+2b\tilde{c}_2+2ba+b^2\tilde{c}_1-ab^3-b^3\right) > 0\\
\iff\ & 2-b-ba-(2\tilde{c}_1+b\tilde{c}_2)+\left(-b^2+2b\tilde{c}_2+b^2\tilde{c}_1\right) > 0\\
\iff\ & 2-b-ba-2\tilde{c}_1+b\tilde{c}_2-b^2+b^2\tilde{c}_1 > 0\\
\iff\ & \left(2-b^2\right)(1-\tilde{c}_1)-b\,(1-\tilde{c}_2+a) > 0\\
\iff\ & -b\,(1-\tilde{c}_2+a) > -\left(2-b^2\right)(1-\tilde{c}_1)\\
\iff\ & -(1-\tilde{c}_2+a) > -\frac{\left(2-b^2\right)}{b}(1-\tilde{c}_1)\\
\iff\ & -(1-\tilde{c}_1-x+a) > -\frac{\left(2-b^2\right)}{b}(1-\tilde{c}_1)\\
\iff\ & x-a > (1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right)
\end{aligned}$$

which holds by Assumption 1 and Lemma A.1, and

$$
\begin{aligned}
& D_2(p_1^N, p_2^N) = \frac{1}{1-b^2}\left[1-b+a-p_2^N+bp_1^N\right] > 0 \\
\iff \quad & 1-b+a-\frac{2-b+2\tilde{c}_2+2a+b\tilde{c}_1-ab^2-b^2}{4-b^2} \\
& \qquad +b\frac{2-b+2\tilde{c}_1-ab+b\tilde{c}_2-b^2}{4-b^2} > 0 \\
\iff \quad & 4-4b+4a-b^2+b^3-b^2a-\left(2-b+2\tilde{c}_2+2a+b\tilde{c}_1-ab^2-b^2\right) \\
& \qquad +\left(2b-b^2+2b\tilde{c}_1-ab^2+b^2\tilde{c}_2-b^3\right) > 0 \\
\iff \quad & 2-b+2a-2\tilde{c}_2+b\tilde{c}_1-b^2-ab^2+b^2\tilde{c}_2 > 0 \\
\iff \quad & \left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right) > 0 \\
\iff \quad & -b\left(1-\tilde{c}_1\right) > -\left(2-b^2\right)\left(1-\tilde{c}_2+a\right). \\
\iff \quad & -\frac{b}{(2-b^2)}\left(1-\tilde{c}_1\right) > -\left(1-\tilde{c}_2+a\right). \\
\iff \quad & -\frac{b}{(2-b^2)}\left(1-\tilde{c}_1\right) > -\left(1-\tilde{c}_1-x+a\right) \\
\iff \quad & \left(1-\tilde{c}_1\right)\left(1-\frac{b}{(2-b^2)}\right) > x-a
\end{aligned}
$$

which holds by Assumption 1 and Lemma A.1.

Substituting (4) and (5) into the manager payoff functions yields:

$$M_1^N = \frac{\left(\left(2-b^2\right)\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}$$

and

$$M_2^N = \frac{\left(\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}.$$

A.2.2 Collusive Phase

Managers collude by setting prices to maximize their joint pay:

$$\max_{p_1,p_2} M_1(p_1,p_2)+M_2(p_1,p_2)$$

which yields collusive prices of $p_1^C = \frac{1+\tilde{c}_1}{2}$ and $p_2^C = \frac{1+\tilde{c}_2+a}{2}$. Collusive payoffs are

$$M_1^C = \frac{1}{1-b^2}\left(\frac{1-\tilde{c}_1}{2}\right)\left(\frac{1-\tilde{c}_1-b\left(1-\tilde{c}_2+a\right)}{2}\right)$$

and

$$M_2^C = \frac{1}{1-b^2}\left(\frac{1-\tilde{c}_2+a}{2}\right)\left(\frac{1-\tilde{c}_2+a-b\left(1-\tilde{c}_1\right)}{2}\right).$$

A.2.3 Defection Phase

When determining payoffs in the defection phase, there are two cases to consider for each manager.

Manager 1:

SubCase A: $\frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} \leq \frac{2b}{2-b^2}$ (Inactive Rival during Defection): If manager 1 defects by setting a price p such that

$$D_2(p, p_2^C) = \frac{bp - \frac{\tilde{c}_2}{2} + \frac{a}{2} - b + \frac{1}{2}}{1-b^2} \leq 0$$

$$bp - \frac{\tilde{c}_2}{2} + \frac{a}{2} - b + \frac{1}{2} \leq 0$$

$$2bp \leq \tilde{c}_2 - a + 2b - 1$$

or

$$p \leq \frac{\tilde{c}_2 - a + 2b - 1}{2b} = 1 + \frac{\tilde{c}_2 - a - 1}{2b} := p_1^{D,I},$$

then its rival is inactive. Additionally, note that manager 1's defection payoff is increasing in p for all $p < p_1^{D,I}$. This is the case because manager 1's payoff when $p \leq p_1^{D,I}$ (and thus, manager 2 is inactive),

$$M_1(p, p_2^C) = (1-p)(p - \tilde{c}_1),$$

is increasing in p for all $p < p_1^M := \frac{1+\tilde{c}_1}{2}$ (the monopoly price) and $p_1^{D,I} < p_1^M$ by

$$\begin{aligned}
& p_1^M > p_1^{D,I} \\
& \frac{1+\tilde{c}_1}{2} > 1 + \frac{\tilde{c}_2 - a - 1}{2b} \\
\iff\quad & 1 + \tilde{c}_1 > 2 + \frac{\tilde{c}_2 - a - 1}{b} \\
\iff\quad & \tilde{c}_1 > 1 + \frac{\tilde{c}_2 - a - 1}{b} \\
\iff\quad & b\tilde{c}_1 > b + \tilde{c}_2 - a - 1 \\
\iff\quad & b\tilde{c}_1 > b + \tilde{c}_1 + x - a - 1 \\
\iff\quad & b\tilde{c}_1 - b - \tilde{c}_1 + 1 > x - a \\
\iff\quad & (1-\tilde{c}_1)(1-b) > x - a
\end{aligned}$$

which holds for all $x-a$ by Lemma A.1. Thus, if a manager sets a defection price such that its rival is inactive, it sets a price just small enough that the rival receives zero demand (i.e., the price $p_1^{D,I} = 1 + \frac{\tilde{c}_2-a-1}{2b}$). In this case, manager 1 receives a payoff of

$$\begin{aligned}
M_1(p_1^{D,I}, p_2^C) &= \left(1 - \left(1 + \frac{\tilde{c}_2 - a - 1}{2b}\right)\right)\left(1 + \frac{\tilde{c}_2 - a - 1}{2b} - \tilde{c}_1\right) \\
&= \frac{1 - \tilde{c}_2 + a}{2b}\left(1 + \frac{\tilde{c}_2 - a - 1}{2b} - \tilde{c}_1\right) \\
&= \frac{1 - \tilde{c}_2 + a}{2b}\left(\frac{2b - 1 + \tilde{c}_2 - a}{2b} - \tilde{c}_1\right)
\end{aligned}$$

when defecting.

Next, we show that, in this case, manager 1 will not choose a defection price such that its rival is active during defection. This occurs if manager 1 sets a price $p > p_1^{D,I}$. Manager 1's payoff when $p > p_1^{D,I}$ is

$$\begin{aligned}
M_1(p, p_2^C) &= D_2(p, p_2^C)\,(p - \tilde{c}_1) \\
&= \frac{1}{1-b^2}\left[1 - b - ba - p + bp_2^C\right](p - \tilde{c}_1).
\end{aligned}$$

Manager 1's payoff is decreasing in p when

$$
\begin{aligned}
\frac{\partial M_1(p, p_2^C)}{\partial p} &= \frac{1}{1-b^2}\left[1-b-ba-p+bp_2^C+\tilde{c}_1\right] < 0 \\
\iff \qquad p &> \frac{1-b-ba+bp_2^C+\tilde{c}_1}{2} \\
\iff \qquad p &> \frac{1-b-ba+b\left(\frac{1+\tilde{c}_2+a}{2}\right)+\tilde{c}_1}{2} \\
\iff \qquad 2p &> 1-b-ba+b\left(\frac{1+\tilde{c}_2+a}{2}\right)+2\tilde{c}_1 \\
\iff \qquad 4p &> 2-2b-2ba+b+b\tilde{c}_2+ba+2\tilde{c}_1 \\
\iff \qquad 4p &> 2-b-ab+b\tilde{c}_2+2\tilde{c}_1.
\end{aligned}
$$

Note that manager 1's payoff is decreasing in p for all $p > p_1^{D,I}$ by

$$
\begin{aligned}
4p_1^{D,I} &\geq 2-b-ab+b\tilde{c}_2+2\tilde{c}_1 \\
\iff \qquad 4+2\frac{\tilde{c}_2-a-1}{b} &\geq 2-b-ab+b\tilde{c}_2+2\tilde{c}_1 \\
\iff \qquad 2+2\frac{\tilde{c}_2-a-1}{b} &\geq -b-ab+b\tilde{c}_2+2\tilde{c}_1 \\
\iff \qquad 2b+2\tilde{c}_2-2a-2 &\geq -b^2-ab^2+b^2\tilde{c}_2+2b\tilde{c}_1 \\
\iff \qquad 2b\left(1-\tilde{c}_1\right) &\geq 2\left(1-\tilde{c}_2+a\right)-b^2\left(1-\tilde{c}_2+a\right) \\
\iff \qquad 2b\left(1-\tilde{c}_1\right) &\geq 2\left(1-\tilde{c}_2+a\right)-b^2\left(1-\tilde{c}_2+a\right) \\
\iff \qquad \frac{2b}{2-b^2} &\geq \frac{1-\tilde{c}_2+a}{1-\tilde{c}_1}
\end{aligned}
$$

which holds by assumption for this case. Thus, defection payoff is increasing in p for $p < p_1^{D,I}$ and decreasing in p for $p > p_1^{D,I}$. This implies the optimal defection price is $p_1^{D,I}$.[2] In summary, it is optimal for manager 1 to charge price $p_1^{D,I}$ and receive a payoff of

$$
M_1(p_1^{D,I}, p_2^C) = \frac{1-\tilde{c}_2+a}{2b}\left(\frac{2b-1+\tilde{c}_2-a}{2b}-\tilde{c}_1\right)
$$

when defecting under this case.

SubCase B: $\frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} > \frac{2b}{2-b^2}$ (Active Rival during Defection): Suppose manager 1 sets a price such that manager 2 receives positive demand during defection. In this case, manager 1's demand is given by $D_1(p_1, p_2) = \frac{1}{1-b^2}\left[1-b-ba-p_1+bp_2\right]$. Manager 1 solves, when defecting,

$$
\begin{aligned}
\max_{p_1} M_1(p, p_2^C) &= \max_{p_1} D_1(p, p_2^C)(p_1-\tilde{c}_1) \\
&= \max_{p_1} \frac{1}{1-b^2}\left[1-b-ba-p_1+bp_2^C\right](p_1-\tilde{c}_1)
\end{aligned}
$$

which yields a defection price of $p_1^{D,A} = \frac{-ab-b+2\tilde{c}_1+b\tilde{c}_2+2}{4}$ and defection profits of

$$
M_1(p_1^{D,A}, p_2^C) = \frac{\left(2\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)\right)^2}{16\left(1-b^2\right)}.
$$

[2]Note that defection profit is continuous in p by Lemma A.7.

It remains to confirm that the rival firm is active during defection. Thus occurs when

$$
\begin{aligned}
& D_2(p_1^{D,A}, p_2^C) = \frac{1}{1-b^2}\left[1-b+a-p_2^C+bp_1^{D,A}\right] > 0 \\
\iff\quad & 1-b+a-\left(\frac{1+\tilde{c}_2+a}{2}\right)+b\left(\frac{-ab-b+2\tilde{c}_1+b\tilde{c}_2+2}{4}\right) > 0 \\
\iff\quad & 4-4b+4a-2-2\tilde{c}_2-2a-b^2a-b^2+2b\tilde{c}_1+b^2\tilde{c}_2+2b > 0 \\
\iff\quad & 2-2b+2a-2\tilde{c}_2-2a-b^2a-b^2+2b\tilde{c}_1+b^2\tilde{c}_2 > 0 \\
\iff\quad & \left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-\left(2b\right)\left(1-\tilde{c}_1\right) > 0 \\
\iff\quad & \frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} > \frac{2b}{2-b^2}
\end{aligned}
$$

which holds by assumption under this case. From SubCase A, recall that a manager never wishes to defect to a price below $p_1^{D,I}$ (as this reduces the manager's payoff). Additionally, any price $p \neq p_1^{D,A}$ such that $p > p_1^{D,I}$ (i.e., the rival manager is active) yields smaller payoff than price $p_1^{D,A}$. Thus, the manager chooses between charging price $p_1^{D,A}$ and price $p_1^{D,I}$. It remains to compare payoff under the two possibilities:

$$
\begin{aligned}
M_1(p_1^{D,I}, p_2^C) &< M_1(p_1^{D,A}, p_2^C) \\
\frac{1-\tilde{c}_2+a}{2b}\left(\frac{2b-1+\tilde{c}_2-a}{2b}-\tilde{c}_1\right) &< \frac{\left(2\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)\right)^2}{16\left(1-b^2\right)} \\
\frac{1-\tilde{c}_2+a}{2b}\left(-\frac{1-\tilde{c}_2+a}{2b}+1-\tilde{c}_1\right) &< \frac{1}{16\left(1-b^2\right)}\left(4\left(1-\tilde{c}_1\right)^2-4b\left(1-\tilde{c}_1\right)\left(1-\tilde{c}_2+a\right)+b^2\left(1-\tilde{c}_2+a\right)^2\right) \\
-\frac{\left(1-\tilde{c}_2+a\right)^2}{4b^2}+\left(1-\tilde{c}_1\right)\left(\frac{1-\tilde{c}_2+a}{2b}\right) &< \frac{1}{16\left(1-b^2\right)}\left(4\left(1-\tilde{c}_1\right)^2-4b\left(1-\tilde{c}_1\right)\left(1-\tilde{c}_2+a\right)+b^2\left(1-\tilde{c}_2+a\right)^2\right) \\
0 &< 4\left(1-\tilde{c}_1\right)^2\frac{1}{16\left(1-b^2\right)}-\left(\frac{4b}{16\left(1-b^2\right)}+\frac{1}{2b}\right)\left(1-\tilde{c}_1\right)\left(1-\tilde{c}_2+a\right) \\
&\quad+\left(\frac{b^2}{16\left(1-b^2\right)}+\frac{1}{4b^2}\right)\left(1-\tilde{c}_2+a\right)^2 \\
0 &< \left(1-\tilde{c}_1\right)^2-\left(b+\frac{2}{b}\left(1-b^2\right)\right)\left(1-\tilde{c}_1\right)\left(1-\tilde{c}_2+a\right) \\
&\quad+\left(\frac{b^2}{4}+\frac{1-b^2}{b^2}\right)\left(1-\tilde{c}_2+a\right)^2 \\
0 &< b\left(1-\tilde{c}_1\right)^2-\left(b^2+2\left(1-b^2\right)\right)\left(1-\tilde{c}_1\right)\left(1-\tilde{c}_2+a\right) \\
&\quad+\left(\frac{b^3}{4}+\frac{1-b^2}{b}\right)\left(1-\tilde{c}_2+a\right)^2 \\
0 &< 4b^2\left(1-\tilde{c}_1\right)^2-4b\left(2-b^2\right)\left(1-\tilde{c}_1\right)\left(1-\tilde{c}_2+a\right)+\left(b^4-4b^2+4\right)\left(1-\tilde{c}_2+a\right)^2 \\
0 &< \left(2b\left(1-\tilde{c}_1\right)-\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)\right)^2
\end{aligned}
$$

Thus, $p = p_1^{D,A}$ is optimal and manager 1 receives a payoff of $M_1(p_1^{D,A}, p_2^C)$ when defecting under this case.

Manager 2:

SubCase A: $\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)} \leq \frac{2b}{2-b^2}$ (Inactive Rival during Defection): If manager 2 defects by setting a price p such that

$$
\begin{aligned}
D_1(p_1^C, p) = \frac{bp-\frac{\tilde{c}_1}{2}-ab-b+\frac{1}{2}}{1-b^2} &\leq 0 \\
2bp-\tilde{c}_1-2ab-2b+1 &\leq 0 \\
2bp &\leq \tilde{c}_1+2ab+2b-1
\end{aligned}
$$

or

$$p \le \frac{\tilde{c}_1 + 2ab + 2b - 1}{2b} = 1 + a + \frac{\tilde{c}_1 - 1}{2b} := p_2^{D,I},$$

then its rival is inactive. Next, note that manager 2's defection payoff is increasing in p for all $p < p_2^{D,I}$. This is the case because manager 2's payoff when $p < p_2^{D,I}$ (thus, manager 1 is inactive),

$$M_2(p_1^C, p) = (1 + a - p)(p - \tilde{c}_2),$$

is increasing in p for all $p < p_2^M$ and $p_2^{D,I} < p_2^M$ by

$$\begin{aligned}
& p_2^M > p_2^{D,I} \\
& \frac{1 + a + \tilde{c}_2}{2} > 1 + a - \frac{1 - \tilde{c}_1}{2b} \\
\iff \quad & 1 + a + \tilde{c}_2 > 2 + 2a - \frac{1 - \tilde{c}_1}{b} \\
\iff \quad & \tilde{c}_2 > 1 + a - \frac{1 - \tilde{c}_1}{b} \\
\iff \quad & \tilde{c}_1 + x > 1 + a - \frac{1 - \tilde{c}_1}{b} \\
\iff \quad & x - a > 1 - \tilde{c}_1 - \frac{1 - \tilde{c}_1}{b} \\
\iff \quad & x - a > (1 - \tilde{c}_1)\left(1 - \frac{1}{b}\right)
\end{aligned}$$

which holds by Lemma A.1 and Assumption 1. Thus, if manager 2 sets a defection price such that the rival is inactive, then they set a price just small enough that the rival receives zero demand (i.e., $p_2^{D,I} = 1 + a - \frac{1 - \tilde{c}_1}{2b}$). In this case, manager 2 receives a payoff of

$$\begin{aligned}
M_2(p_1^C, p_2^{D,I}) &= \left(1 + a - \left(1 + a - \frac{1 - \tilde{c}_1}{2b}\right)\right)\left(\left(1 + a - \frac{1 - \tilde{c}_1}{2b}\right) - \tilde{c}_2\right) \\
&= \frac{1 - \tilde{c}_1}{2b}\left(\left(1 + a - \frac{1 - \tilde{c}_1}{2b}\right) - \tilde{c}_2\right) \\
&= \frac{1 - \tilde{c}_1}{2b}\left(\left(\frac{2b + 2ba + \tilde{c}_1 - 1}{2b}\right) - \tilde{c}_2\right).
\end{aligned}$$

Next, we show that, in this case, manager 2 will not choose a defection price such that its rival is active during defection. This occurs if manager 2 sets a price $p > p_2^{D,I}$. Manager 2's payoff when $p > p_2^{D,I}$ is

$$\begin{aligned}
M_2(p_1^C, p) &= D_2(p_1^C, p)\,(p - \tilde{c}_2) \\
&= \frac{1}{1 - b^2}\left[1 - b + a - p + bp_1^C\right](p - \tilde{c}_2).
\end{aligned}$$

Manager 2's payoff is decreasing in p when

$$\begin{aligned}
& \frac{\partial M_2(p_1^C, p)}{\partial p} = \frac{1}{1 - b^2}\left[1 - b + a - 2p + bp_1^C + \tilde{c}_2\right] < 0 \\
\iff \quad & p > \frac{1 - b + a + bp_1^C + \tilde{c}_2}{2} \\
\iff \quad & p > \frac{1 - b + a + b\left(\frac{1 + \tilde{c}_1}{2}\right) + \tilde{c}_2}{2} \\
\iff \quad & 2p > 1 - b + a + b\left(\frac{1 + \tilde{c}_1}{2}\right) + 2\tilde{c}_2 \\
\iff \quad & 4p > 2 - 2b + 2a + b + b\tilde{c}_1 + 2\tilde{c}_2 \\
\iff \quad & 4p > 2 - b + 2a + b\tilde{c}_1 + 2\tilde{c}_2.
\end{aligned}$$

Note that manager 2's payoff is decreasing in p for all $p > p_2^{D,I}$ by

$$\begin{aligned}
& & 4p_2^{D,I} &\geq 2 - b + 2a + b\tilde{c}_1 + 2\tilde{c}_2 \\
&\iff & 4 + 4a - 2\frac{1-\tilde{c}_1}{b} &\geq 2 - b + 2a + b\tilde{c}_1 + 2\tilde{c}_2 \\
&\iff & 2 + 2a - 2\frac{1-\tilde{c}_1}{b} &\geq -b + b\tilde{c}_1 + 2\tilde{c}_2 \\
&\iff & 2b + 2ab - 2 + 2\tilde{c}_1 &\geq -b^2 + b^2\tilde{c}_1 + 2b\tilde{c}_2 \\
&\iff & 2b + 2ab - 2b\tilde{c}_2 &\geq 2 - 2\tilde{c}_1 - b^2 + b^2\tilde{c}_1 \\
&\iff & 2b + 2ab - 2b\tilde{c}_2 &\geq 2 - 2\tilde{c}_1 - b^2 + b^2\tilde{c}_1 \\
&\iff & 2b\left(1 - \tilde{c}_2 + a\right) &\geq \left(2 - b^2\right)\left(1 - \tilde{c}_1\right) \\
&\iff & \frac{2b}{2-b^2} &\geq \frac{1-\tilde{c}_1}{1-\tilde{c}_2+a}
\end{aligned}$$

which holds by assumption for this case. Thus, defection payoff is increasing in p for $p < p_2^{D,I}$ and decreasing in p for $p > p_2^{D,I}$. This implies the optimal defection price is $p_2^{D,I}$.[3] In summary, it is optimal for manager 2 to charge price $p_2^{D,I}$ and receive a payoff of

$$M_2(p_1^C, p_2^{D,I}) = \frac{1-\tilde{c}_1}{2b}\left(\left(\frac{2b + 2ba + \tilde{c}_1 - 1}{2b}\right) - \tilde{c}_2\right)$$

when defecting under this case.

SubCase B: $\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)} > \frac{2b}{2-b^2}$ (Active Rival during Defection): Suppose manager 2 sets a price such that manager 1 receives positive demand during defection. In this case, manager 2's demand is given by $D_2(p_1, p_2) = \frac{1}{1-b^2}\left[1 - b + a - p_2 + bp_1\right]$. Manager 2 solves, when defecting,

$$\begin{aligned}
\max_{p_2} M_2(p_1^C, p_2) &= \max_{p_2} D_2(p_1^C, p_2)(p_2 - \tilde{c}_2) \\
&= \max_{p_2} \frac{1}{1-b^2}\left[1 - b + a - p_2 + bp_1^C\right](p_2 - \tilde{c}_2)
\end{aligned}$$

which yields a defection price of $p_2^{D,A} = \frac{2a - b + 2\tilde{c}_2 + b\tilde{c}_1 + 2}{4}$ and defection profits of

$$M_2(p_1^C, p_2^{D,A}) = \frac{\left(2\left(1 - \tilde{c}_2 + a\right) - b\left(1 - \tilde{c}_1\right)\right)^2}{16\left(1-b^2\right)}.$$

It remains to confirm that the rival firm is active during defection. Thus occurs when

$$\begin{aligned}
& & D_1(p_1^C, p_2^{D,A}) = \frac{1}{1-b^2}\left[1 - b - ba - p_1^C + bp_2^D\right] &> 0 \\
&\iff & \frac{-2b - 2\tilde{c}_1 - 2ab + 2b\tilde{c}_2 + b^2\tilde{c}_1 - b^2 + 2}{4\left(1-b^2\right)} &> 0 \\
&\iff & \frac{\left(2-b^2\right)\left(1-\tilde{c}_1\right) - 2b\left(1 - \tilde{c}_2 + a\right)}{4\left(1-b^2\right)} &> 0 \\
&\iff & \left(2-b^2\right)\left(1-\tilde{c}_1\right) - 2b\left(1 - \tilde{c}_2 + a\right) &> 0 \\
&\iff & \frac{1-\tilde{c}_1}{1+a-\tilde{c}_2} &> \frac{2b}{2-b^2}
\end{aligned}$$

which holds by assumption under this case. From SubCase A, recall that a manager never wishes to defect to a price below $p_2^{D,I}$ (as this reduces payoff). Additionally, any price $p \neq p_2^{D,A}$ such that $p > p_2^{D,I}$ (i.e., the

[3] Note that defection profit is continuous in p by Lemma A.7.

rival manager is active) yields smaller payoff than price $p_2^{D,A}$. Thus, the manager chooses between charging price $p_2^{D,A}$ and price $p_2^{D,I}$. It remains to compare the payoff under the two possibilities:

$$
\begin{aligned}
M_2(p_1^C, p_2^{D,I}) &< M_2(p_1^C, p_2^{D,A}) \\
\left(\frac{1-\tilde{c}_1}{2b}\right)\left(\frac{2b+2ab+\tilde{c}_1-1}{2b}-\tilde{c}_2\right) &< \frac{\left(2\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)\right)^2}{16\left(1-b^2\right)} \\
-\left(\frac{1-\tilde{c}_1}{2b}\right)^2+\frac{1-\tilde{c}_1}{2b}\left(1+a-\tilde{c}_2\right) &< \frac{1}{16}\frac{1}{1-b^2}\left(4\left(1-\tilde{c}_2+a\right)^2-4b\left(1-\tilde{c}_1\right)\left(1-\tilde{c}_2+a\right)+b^2\left(1-\tilde{c}_1\right)^2\right) \\
-\left(\frac{1-\tilde{c}_1}{2b}\right)^2+\frac{1-\tilde{c}_1}{2b}\left(1+a-\tilde{c}_2\right) &< \frac{1}{4}\frac{1}{1-b^2}\left(1-\tilde{c}_2+a\right)^2-\frac{1}{4}\frac{b}{1-b^2}\left(1-\tilde{c}_1\right)\left(1-\tilde{c}_2+a\right)+\frac{b^2}{16}\frac{1}{1-b^2}\left(1-\tilde{c}_1\right)^2 \\
0 &< \frac{1}{4}\frac{1}{1-b^2}\left(1-\tilde{c}_2+a\right)^2-\left(\frac{1}{4}\frac{b}{1-b^2}+\frac{1}{2b}\right)\left(1-\tilde{c}_1\right)\left(1-\tilde{c}_2+a\right) \\
&\quad+\left(\frac{b^2}{16}\frac{1}{1-b^2}+\frac{1}{4b^2}\right)\left(1-\tilde{c}_1\right)^2 \\
0 &< \frac{1}{1-b^2}\left(1-\tilde{c}_2+a\right)^2-\left(\frac{b}{1-b^2}+\frac{2}{b}\right)\left(1-\tilde{c}_1\right)\left(1-\tilde{c}_2+a\right) \\
&\quad+\left(\frac{b^2}{4}\frac{1}{1-b^2}+\frac{1}{b^2}\right)\left(1-\tilde{c}_1\right)^2 \\
0 &< b\left(1-\tilde{c}_2+a\right)^2-\left(b^2+2\left(1-b^2\right)\right)\left(1-\tilde{c}_1\right)\left(1-\tilde{c}_2+a\right) \\
&\quad+\left(\frac{b^3}{4}+\frac{1-b^2}{b}\right)\left(1-\tilde{c}_1\right)^2 \\
0 &< b\left(1-\tilde{c}_2+a\right)^2-2\left(\frac{b^2}{2}+\left(1-b^2\right)\right)\left(1-\tilde{c}_1\right)\left(1-\tilde{c}_2+a\right) \\
&\quad+\left(\frac{b^3}{4}+\frac{1-b^2}{b}\right)\left(1-\tilde{c}_1\right)^2 \\
0 &< 4b^2\left(1-\tilde{c}_2+a\right)^2-2\left(2b^3+4b\left(1-b^2\right)\right)\left(1-\tilde{c}_1\right)\left(1-\tilde{c}_2+a\right) \\
&\quad+\left(b^4+4\left(1-b^2\right)\right)\left(1-\tilde{c}_1\right)^2 \\
0 &< 4b^2\left(1-\tilde{c}_2+a\right)^2-2\left(2b\right)\left(b^2+2\left(1-b^2\right)\right)\left(1-\tilde{c}_1\right)\left(1-\tilde{c}_2+a\right) \\
&\quad+\left(2-b^2\right)^2\left(1-\tilde{c}_1\right)^2 \\
0 &< 4b^2\left(1-\tilde{c}_2+a\right)^2-2\left(2b\right)\left(2-b^2\right)\left(1-\tilde{c}_1\right)\left(1-\tilde{c}_2+a\right)+\left(2-b^2\right)^2\left(1-\tilde{c}_1\right)^2 \\
0 &< \left(2b\left(1-\tilde{c}_2+a\right)-\left(2-b^2\right)\left(1-\tilde{c}_1\right)\right)^2
\end{aligned}
$$

Thus, $p_2^{D,A}$ is the optimal defection price and manager 2 receives a payoff of $M_2(p_1^C, p_2^{D,A})$ when defecting in this case.

A.3 Critical Discount Factor

Manager 1:

SubCase A ($\frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} > \frac{2b}{2-b^2}$): When $\frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} > \frac{2b}{2-b^2}$, the critical discount factor of manager 1 is[4]

$$\begin{aligned}
\delta_1^* &= \frac{M_1(p_1^{D,A}, p_2^C) - M_1(p_1^C, p_2^C)}{M_1(p_1^{D,A}, p_2^C) - M_1(p_1^N, p_2^N)} \\
&= \frac{\frac{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{16(1-b^2)} - \frac{1}{1-b^2}\left(\frac{1-\tilde{c}_1}{2}\right)\left(\frac{1-\tilde{c}_1-b(1-\tilde{c}_2+a)}{2}\right)}{\frac{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{16(1-b^2)} - \frac{((2-b^2)(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{(4-b^2)^2(1-b^2)}} \\
&= \frac{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2 - 16\left(\frac{1-\tilde{c}_1}{2}\right)\left(\frac{1-\tilde{c}_1-b(1-\tilde{c}_2+a)}{2}\right)}{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2 - 16\frac{((2-b^2)(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{(4-b^2)^2}} \\
&= \frac{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2 - 4(1-\tilde{c}_1)(1-\tilde{c}_1-b(1-\tilde{c}_2+a))}{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2 - 16\frac{((2-b^2)(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{(4-b^2)^2}} \\
&= \frac{4(1-\tilde{c}_1)^2 + b^2(1-\tilde{c}_2+a)^2 - 4b(1-\tilde{c}_1)(1-\tilde{c}_2+a) - 4(1-\tilde{c}_1)(1-\tilde{c}_1-b(1-\tilde{c}_2+a))}{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2 - 16\frac{((2-b^2)(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{(4-b^2)^2}} \\
&= \frac{b^2(1-\tilde{c}_2+a)^2}{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2 - 16\frac{((2-b^2)(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{(4-b^2)^2}}
\end{aligned}$$

SubCase B ($\frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} \leq \frac{2b}{2-b^2}$): The critical discount factor of manager 1 is[5]

$$\begin{aligned}
\delta_1^* &= \frac{M_1(p_1^{D,I}, p_2^C) - M_1(p_1^C, p_2^C)}{M_1(p_1^{D,I}, p_2^C) - M_1(p_1^N, p_2^N)} \\
&= \frac{\frac{1-\tilde{c}_2+a}{2b}\left(\frac{2b-1+\tilde{c}_2-a}{2b} - \tilde{c}_1\right) - \frac{1}{1-b^2}\left(\frac{1-\tilde{c}_1}{2}\right)\left(\frac{1-\tilde{c}_1-b(1-\tilde{c}_2+a)}{2}\right)}{\frac{1-\tilde{c}_2+a}{2b}\left(\frac{2b-1+\tilde{c}_2-a}{2b} - \tilde{c}_1\right) - \frac{((2-b^2)(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{(4-b^2)^2(1-b^2)}}.
\end{aligned}$$

In summary, the critical discount factor of manager 1 is

$$\delta_1^* = \begin{cases} \frac{\frac{1-\tilde{c}_2+a}{2b}\left(\frac{2b-1+\tilde{c}_2-a}{2b} - \tilde{c}_1\right) - \frac{1}{1-b^2}\left(\frac{1-\tilde{c}_1}{2}\right)\left(\frac{1-\tilde{c}_1-b(1-\tilde{c}_2+a)}{2}\right)}{\frac{1-\tilde{c}_2+a}{2b}\left(\frac{2b-1+\tilde{c}_2-a}{2b} - \tilde{c}_1\right) - \frac{((2-b^2)(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{(4-b^2)^2(1-b^2)}} & \frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} \leq \frac{2b}{2-b^2} \\ \frac{b^2(1-\tilde{c}_2+a)^2}{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2 - 16\frac{((2-b^2)(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{(4-b^2)^2}} & \frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} > \frac{2b}{2-b^2} \end{cases}.$$

Manager 2:

[4] $p_1^{D,A} = \frac{-ab-b+2\tilde{c}_1+b\tilde{c}_2+2}{4}$ and is derived in Section A.2.3.
[5] $p_1^{D,I} = 1 + \frac{\tilde{c}_2-a-1}{2b}$ and is derived in Section A.2.3.

SubCase A ($\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)} > \frac{2b}{2-b^2}$): The critical discount factor of manager 2 is[6]

$$\begin{aligned}
\delta_2^* &= \frac{M_2(p_1^C, p_2^{D,A}) - M_2(p_1^C, p_2^C)}{M_2(p_1^C, p_2^{D,A}) - M_2(p_1^N, p_2^N)} \\
&= \frac{\frac{(2(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{16(1-b^2)} - \frac{1}{1-b^2}\left(\frac{1-\tilde{c}_2+a}{2}\right)\left(\frac{1-\tilde{c}_2+a-b(1-\tilde{c}_1)}{2}\right)}{\frac{(2(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{16(1-b^2)} - \frac{((2-b^2)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{(4-b^2)^2(1-b^2)}} \\
&= \frac{(2(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2 - 16\left(\frac{1-\tilde{c}_2+a}{2}\right)\left(\frac{1-\tilde{c}_2+a-b(1-\tilde{c}_1)}{2}\right)}{(2(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2 - 16\frac{((2-b^2)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{(4-b^2)^2}} \\
&= \frac{(2(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2 - 4(1-\tilde{c}_2+a)(1-\tilde{c}_2+a-b(1-\tilde{c}_1))}{(2(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2 - 16\frac{((2-b^2)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{(4-b^2)^2}} \\
&= \frac{4(1-\tilde{c}_2+a)^2 - 4b(1-\tilde{c}_2+a)(1-\tilde{c}_1) + b^2(1-\tilde{c}_1)^2 - 4(1-\tilde{c}_2+a)(1-\tilde{c}_2+a-b(1-\tilde{c}_1))}{(2(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2 - 16\frac{((2-b^2)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{(4-b^2)^2}} \\
&= \frac{b^2(1-\tilde{c}_1)^2}{(2(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2 - 16\frac{((2-b^2)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{(4-b^2)^2}}
\end{aligned}$$

SubCase B ($\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)} \leq \frac{2b}{2-b^2}$): The critical discount factor of manager 2 is[7]

$$\begin{aligned}
\delta_2^* &= \frac{M_2(p_1^C, p_2^{D,I}) - M_2(p_1^C, p_2^C)}{M_2(p_1^C, p_2^{D,I}) - M_2(p_1^N, p_2^N)} \\
&= \frac{\left(\frac{1-\tilde{c}_1}{2b}\right)\left(\frac{2b+2ab+\tilde{c}_1-1}{2b} - \tilde{c}_2\right) - \frac{1}{1-b^2}\left(\frac{1-\tilde{c}_2+a}{2}\right)\left(\frac{1-\tilde{c}_2+a-b(1-\tilde{c}_1)}{2}\right)}{\left(\frac{1-\tilde{c}_1}{2b}\right)\left(\frac{2b+2ab+\tilde{c}_1-1}{2b} - \tilde{c}_2\right) - \frac{((2-b^2)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{(4-b^2)^2(1-b^2)}}.
\end{aligned}$$

In summary, the critical discount factor of manager 2 is

$$\delta_2^* = \begin{cases} \frac{\left(\frac{1-\tilde{c}_1}{2b}\right)\left(\frac{2b+2ab+\tilde{c}_1-1}{2b} - \tilde{c}_2\right) - \frac{1}{1-b^2}\left(\frac{1-\tilde{c}_2+a}{2}\right)\left(\frac{1-\tilde{c}_2+a-b(1-\tilde{c}_1)}{2}\right)}{\left(\frac{1-\tilde{c}_1}{2b}\right)\left(\frac{2b+2ab+\tilde{c}_1-1}{2b} - \tilde{c}_2\right) - \frac{((2-b^2)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{(4-b^2)^2(1-b^2)}} & \frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)} \leq \frac{2b}{2-b^2} \\ \frac{b^2(1-\tilde{c}_1)^2}{(2(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2 - 16\frac{((2-b^2)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{(4-b^2)^2}} & \frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)} > \frac{2b}{2-b^2} \end{cases}.$$

Lemma A.6. *i)* $\delta_1^*(\theta_1,\theta_2) = \delta_2^*(\theta_1,\theta_2)$ *when* $x-a=0$ *,*
ii) $\delta_2^*(\theta_1,\theta_2) > \delta_1^*(\theta_1,\theta_2)$ *when* $x-a>0$*, and*
iii) $\delta_2^*(\theta_1,\theta_2) < \delta_1^*(\theta_1,\theta_2)$ *when* $x-a<0$.

Proof. Part i) $\delta_1^*(\theta_1,\theta_2) = \delta_2^*(\theta_1,\theta_2)$ when $x-a=0$ because firms are effectively symmetric.

Let

$$\delta_{sym}^* = \max\{\delta_1^*(\theta_1,\theta_2), \delta_2^*(\theta_1,\theta_2)\} = \delta_1^*(\theta_1,\theta_2) = \delta_2^*(\theta_1,\theta_2)$$

denote the critical discount factor when $x-a=0$.

Part ii) Consider the case of $x-a>0$. By Lemma A.4 and Lemma A.5, $\delta_1^*(\theta_1,\theta_2)$ is decreasing in $x-a$ and, thus, $\delta_1^*(\theta_1,\theta_2) < \delta_{sym}^*$. By Lemma A.2 and Lemma A.3, $\delta_2^*(\theta_1,\theta_2)$ is increasing in $x-a$ and, thus, $\delta_2^*(\theta_1,\theta_2) > \delta_{sym}^*$. Therefore, $\delta_2^*(\theta_1,\theta_2) > \delta_{sym}^* > \delta_1^*(\theta_1,\theta_2)$.

Part iii) Consider the case of $x-a<0$. By Lemma A.4 and Lemma A.5, $\delta_1^*(\theta_1,\theta_2)$ is decreasing in $x-a$ and, thus, $\delta_1^*(\theta_1,\theta_2) > \delta_{sym}^*$. By Lemma A.2 and Lemma A.3, $\delta_2^*(\theta_1,\theta_2)$ is increasing in $x-a$ and, thus, $\delta_2^*(\theta_1,\theta_2) < \delta_{sym}^*$. Therefore, $\delta_2^*(\theta_1,\theta_2) < \delta_{sym}^* < \delta_1^*(\theta_1,\theta_2)$. ∎

[6] $p_2^{D,A} = \frac{2a-b+2\tilde{c}_2+b\tilde{c}_1+2}{4}$ and is derived in Section A.2.3.
[7] $p_2^{D,I} = 1+a+\frac{\tilde{c}_1-1}{2b}$ and is derived in Section A.2.3.

By Lemma A.6, the critical discount factor equals

$$\delta^* = \begin{cases} \dfrac{b^2(1-\tilde{c}_2+a)^2}{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2-16\frac{((2-b^2)(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{(4-b^2)^2}} & \text{if } \frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} > \frac{2b}{2-b^2} \text{ and } x-a \leq 0 \\ \dfrac{\frac{1-\tilde{c}_2+a}{2b}\left(\frac{2b-1+\tilde{c}_2-a}{2b}-\tilde{c}_1\right)-\frac{1}{1-b^2}\left(\frac{1-\tilde{c}_1}{2}\right)\left(\frac{1-\tilde{c}_1-b(1-\tilde{c}_2+a)}{2}\right)}{\frac{1-\tilde{c}_2+a}{2b}\left(\frac{2b-1+\tilde{c}_2-a}{2b}-\tilde{c}_1\right)-\frac{((2-b^2)(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{(4-b^2)^2(1-b^2)}} & \text{if } \frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} \leq \frac{2b}{2-b^2} \text{ and } x-a \leq 0 \\ \dfrac{b^2(1-\tilde{c}_1)^2}{(2(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2-16\frac{((2-b^2)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{(4-b^2)^2}} & \text{if } \frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)} > \frac{2b}{2-b^2} \text{ and } x-a > 0 \\ \dfrac{\left(\frac{1-\tilde{c}_1}{2b}\right)\left(\frac{2b+2ab+\tilde{c}_1-1}{2b}-\tilde{c}_2\right)-\frac{1}{1-b^2}\left(\frac{1-\tilde{c}_2+a}{2}\right)\left(\frac{1-\tilde{c}_2+a-b(1-\tilde{c}_1)}{2}\right)}{\left(\frac{1-\tilde{c}_1}{2b}\right)\left(\frac{2b+2ab+\tilde{c}_1-1}{2b}-\tilde{c}_2\right)-\frac{((2-b^2)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{(4-b^2)^2(1-b^2)}} & \text{if } \frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)} \leq \frac{2b}{2-b^2} \text{ and } x-a > 0. \end{cases}$$

Note that when compensation structures are identical (i.e., $\theta_1 = \theta_2$) and firms are homogenous (i.e., $c_1 = c_2$ and $a = 0$), the critical discount factor is

$$\delta^*_{sym} = \begin{cases} \frac{(2-b)^2}{b^2-8b+8} & \text{if } \frac{2b}{2-b^2} \leq 1 \\ \frac{(2-b)^2\left(1-b-b^2\right)}{4-8b+b^2+3b^3-2b^4} & \text{if } \frac{2b}{2-b^2} > 1 \end{cases}$$

or

$$\delta^*_{sym} = \begin{cases} \frac{(2-b)^2}{b^2-8b+8} & \text{if } b \leq -1+\sqrt{3} \\ \frac{(2-b)^2\left(1-b-b^2\right)}{4-8b+b^2+3b^3-2b^4} & \text{if } b > -1+\sqrt{3} \end{cases}.$$

Lemma A.7. *δ^* is continuous in x and $\tilde{c}_1$.*

Proof. It suffices to show that manager payoffs under each phase (competition, collusion and defection) are continuous. Manager payoffs under Nash competition and collusion are clearly continuous in x and $\tilde{c}_1$. It remains to show that defection payoffs are continuous at the point where the rival firm becomes inactive when a manager defects ($\frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} = \frac{2b}{2-b^2}$ when manager 1 defects and $\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)} = \frac{2b}{2-b^2}$ when manager 2 defects).

Manager 1: We wish to show

$$\left(\frac{1-\tilde{c}_2+a}{2b}\right)\left(\frac{2b-a+\tilde{c}_2-1}{2b}-\tilde{c}_1\right) = \frac{\left(2\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)\right)^2}{16\left(1-b^2\right)}$$

when $\frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} = \frac{2b}{2-b^2}$. Note that $\frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} = \frac{2b}{2-b^2}$ holds if and only if $\frac{1-(\tilde{c}_2-a)}{2b} = \frac{1-\tilde{c}_1}{2-b^2}$. Simplifying the left hand side yields

$$\begin{aligned} \left(\frac{1-\tilde{c}_2+a}{2b}\right)\left(1-\frac{1-\tilde{c}_2+a}{2b}-\tilde{c}_1\right) &= \left(\frac{1-\tilde{c}_1}{2-b^2}\right)\left(1-\tilde{c}_1-\frac{1-\tilde{c}_1}{2-b^2}\right) \\ &= \left(1-\tilde{c}_1\right)^2\frac{1}{2-b^2}\left(1-\frac{1}{2-b^2}\right) \\ &= \left(1-\tilde{c}_1\right)^2\frac{1}{2-b^2}\left(\frac{1-b^2}{2-b^2}\right) \end{aligned}$$

Substituting into the right hand side yields

$$\begin{aligned}
\frac{\left(2\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)\right)^2}{16\left(1-b^2\right)} &= \frac{\left(2\left(1-\tilde{c}_1\right)-\left(1-\tilde{c}_1\right)\frac{2b^2}{2-b^2}\right)^2}{16\left(1-b^2\right)} \\
&= \frac{\left(\left(1-\tilde{c}_1\right)\left(2-\frac{2b^2}{2-b^2}\right)\right)^2}{16\left(1-b^2\right)} \\
&= \frac{\left(\left(1-\tilde{c}_1\right)\left(\frac{4-2b^2-2b^2}{2-b^2}\right)\right)^2}{16\left(1-b^2\right)} \\
&= \frac{\left(\left(1-\tilde{c}_1\right)\left(4\frac{1-b^2}{2-b^2}\right)\right)^2}{16\left(1-b^2\right)} \\
&= \left(16\left(1-b^2\right)^2\right)\frac{1}{\left(2-b^2\right)^2}\frac{\left(1-\tilde{c}_1\right)^2}{16\left(1-b^2\right)} \\
&= \frac{1-b^2}{\left(2-b^2\right)^2}\left(1-\tilde{c}_1\right)^2
\end{aligned}$$

and the proof is complete.

Manager 2: We wish to show

$$\left(\frac{1-\tilde{c}_1}{2b}\right)\left(1+a-\frac{1-\tilde{c}_1}{2b}-\tilde{c}_2\right) = \frac{\left(2\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)\right)^2}{16\left(1-b^2\right)}$$

when $\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)} = \frac{2b}{2-b^2}$. Note that $\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)} = \frac{2b}{2-b^2}$ holds if and only if $\frac{1-\tilde{c}_1}{2b} = \frac{1-\tilde{c}_2+a}{2-b^2}$. Substituting this equality into the left hand side yields:

$$\begin{aligned}
\left(\frac{1-\tilde{c}_2+a}{2-b^2}\right)\left(1+a-\frac{1-\tilde{c}_2+a}{2-b^2}-\tilde{c}_2\right) &= \left(\frac{1-\tilde{c}_2+a}{2-b^2}\right)\left(1-\tilde{c}_2+a\right)\left(1-\frac{1}{2-b^2}\right) \\
&= \left(\frac{1-\tilde{c}_2+a}{2-b^2}\right)\left(1-\tilde{c}_2+a\right)\left(\frac{2-b^2-1}{2-b^2}\right) \\
&= \left(\frac{1-\tilde{c}_2+a}{2-b^2}\right)\left(1-\tilde{c}_2+a\right)\left(\frac{1-b^2}{2-b^2}\right) \\
&= \frac{\left(1-b^2\right)}{\left(2-b^2\right)^2}\left(1-\tilde{c}_2+a\right)^2
\end{aligned}$$

Substituting the equality into the right hand side yields

$$
\begin{aligned}
\frac{\left(2\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)\right)^2}{16\left(1-b^2\right)} &= \frac{\left(2\left(1-\tilde{c}_2+a\right)-b\frac{2b}{2-b^2}\left(1-\tilde{c}_2+a\right)\right)^2}{16\left(1-b^2\right)} \\
&= \frac{\left(2\left(1-\tilde{c}_2+a\right)-\frac{2b^2}{2-b^2}\left(1-\tilde{c}_2+a\right)\right)^2}{16\left(1-b^2\right)} \\
&= \frac{\left(\left(1-\tilde{c}_2+a\right)\left(\frac{4-2b^2-2b^2}{2-b^2}\right)\right)^2}{16\left(1-b^2\right)} \\
&= \frac{\left(\left(1-\tilde{c}_2+a\right)\left(\frac{4-4b^2}{2-b^2}\right)\right)^2}{16\left(1-b^2\right)} \\
&= \left(1-b^2\right)\left(\left(1-\tilde{c}_2+a\right)\left(\frac{1}{2-b^2}\right)\right)^2 \\
&= \frac{\left(1-b^2\right)}{\left(2-b^2\right)^2}\left(1-\tilde{c}_2+a\right)^2
\end{aligned}
$$

and the proof is complete. □

A.4 Assumptions

In the main text, we assume Assumption 1. In the following Lemma, we show that this assumption ensures that $\delta^*(\theta_1, \theta_2) < 1$.

Lemma A.8. *$\delta^*(\theta_1, \theta_2) < 1$ if and only if Assumption 1 holds.*

Proof. ($\Longleftarrow$)Assume Assumption 1 holds. Collusion is sustainable for some discount factor less than 1 if

$$
\frac{M_i^C(\theta_1, \theta_2)}{1-\delta} \geq M_i^D(\theta_1, \theta_2) + \delta\frac{M_i^N(\theta_1, \theta_2)}{1-\delta}.
$$

If $M_1^C(\theta_1, \theta_2) > M_1^N(\theta_1, \theta_2)$ and $M_2^C(\theta_1, \theta_2) > M_2^N(\theta_1, \theta_2)$, then the above inequality is satisfied for some δ sufficiently close to 1 for both managers. It remains to show Assumption 1 implies $M_1^C(\theta_1, \theta_2) > M_1^N(\theta_1, \theta_2)$ and $M_2^C(\theta_1, \theta_2) > M_2^N(\theta_1, \theta_2)$. Note that

$$
\begin{aligned}
& M_1^C(\theta_1, \theta_2) > M_1^N(\theta_1, \theta_2) \\
\Longleftrightarrow \quad & \frac{1}{1-b^2}\left(\frac{1-\tilde{c}_1}{2}\right)\left(\frac{1-\tilde{c}_1-b\left(1-\tilde{c}_2+a\right)}{2}\right) > \frac{\left(\left(2-b^2\right)\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)} \\
\Longleftrightarrow \quad & x-a > \left(1-\tilde{c}_1\right)\left(1-\frac{\left(4-b^2\right)\sqrt{b^2+8}-b^3}{8}\right) \\
\text{and} \quad & x-a < \left(1-\tilde{c}_1\right)\left(1+\frac{\left(4-b^2\right)\sqrt{b^2+8}+b^3}{8}\right).
\end{aligned}
$$

The first inequality holds by Assumption 1. The second inequality holds trivially as $\tilde{c}_2 < 1+a \implies x-a <$

$1-\tilde{c}_1$. Note that

$$
\begin{aligned}
& M_2^C(\theta_1,\theta_2) > M_2^N(\theta_1,\theta_2) \\
\iff \quad & \frac{1}{1-b^2}\left(\frac{1-\tilde{c}_2+a}{2}\right)\left(\frac{1-\tilde{c}_2+a-b\left(1-\tilde{c}_1\right)}{2}\right) > \\
& \frac{\left(\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)} \\
\iff \quad & \left(1-\tilde{c}_1\right)\left(1-\frac{\left(4-b^2\right)\sqrt{b^2+8}+b^3}{16-6b^2}\right) > x-a \\
\text{or} \quad & \left(1-\tilde{c}_1\right)\left(1+\frac{\left(4-b^2\right)\sqrt{b^2+8}-b^3}{16-6b^2}\right) < x-a.
\end{aligned}
$$

The first inequality holds by Assumption 1. The second inequality never holds by $\tilde{c}_2 < 1+a$.[8] Thus, Assumption 1 implies $M_1^C(\theta_1,\theta_2) > M_1^N(\theta_1,\theta_2)$ and $M_2^C(\theta_1,\theta_2) > M_2^N(\theta_1,\theta_2)$.

($\implies$) Assume $\delta^*(\theta_1,\theta_2) < 1$. $\delta^*(\theta_1,\theta_2) < 1$ implies that there exists a $\delta \in (\delta^*(\theta_1,\theta_2), 1)$ such that

$$\frac{M_i^C(\theta_1,\theta_2)}{1-\delta} \geq M_i^D(\theta_1,\theta_2) + \delta\frac{M_i^N(\theta_1,\theta_2)}{1-\delta}$$

for both manager 1 and 2. Note that $M_i^D(\theta_1,\theta_2) > M_i^C(\theta_1,\theta_2)$. Thus, the above inequality implies $M_1^C(\theta_1,\theta_2) > M_1^N(\theta_1,\theta_2)$ and $M_2^C(\theta_1,\theta_2) > M_2^N(\theta_1,\theta_2)$ hold. As shown in the first part of the proof,

$$M_1^C(\theta_1,\theta_2) > M_1^N(\theta_1,\theta_2) \implies x-a > \left(1-\tilde{c}_1\right)\left(1-\frac{\left(4-b^2\right)\sqrt{b^2+8}-b^3}{8}\right)$$

and

$$M_2^C(\theta_1,\theta_2) > M_2^N(\theta_1,\theta_2) \implies x-a < \left(1-\tilde{c}_1\right)\left(1-\frac{\left(4-b^2\right)\sqrt{b^2+8}+b^3}{16-6b^2}\right).$$

Thus, Assumption 1 holds. □

A.5 Proofs from Main Text

In this section, we provide additional details behind the proofs of the results in the main text. As in the text appendix, let $\delta_i^*(\tilde{c}_1,x,a)$ denote the critical discount factor of manager i when the perceived marginal cost of manager 1 is $\tilde{c}_1$, the asymmetry in perceived marginal cost is $x=\tilde{c}_2-\tilde{c}_1$ and the asymmetry in product quality is a. The industry critical discount factor $\delta^*(\tilde{c}_1,x,a)$ is defined analogously.

A.5.1 Proof of Proposition 1: Additional Details

Proposition (Proposition 1 from the Main Text). *i)* $\delta_1^*(\theta_1,\theta_2) = \delta_2^*(\theta_1,\theta_2)$ *when* $\tilde{c}_2-\tilde{c}_1-a=0$,
ii) $\delta_2^*(\theta_1,\theta_2) > \delta_1^*(\theta_1,\theta_2)$ *when* $\tilde{c}_2-\tilde{c}_1-a>0$, *and*
iii) $\delta_2^*(\theta_1,\theta_2) < \delta_1^*(\theta_1,\theta_2)$ *when* $\tilde{c}_2-\tilde{c}_1-a<0$.

Proof. See the proof of Lemma A.6. □

A.5.2 Proof of Lemma 1: Additional Details

Lemma (Lemma 1 from the Text Appendix). *Suppose* $x-a\neq 0$. *Then,* $\delta^*(\tilde{c}_1,x,a)$ *is increasing in* $\tilde{c}_1$.

[8] $\tilde{c}_2 < 1+a \implies x-a < 1-\tilde{c}_1 < \left(1-\tilde{c}_1\right)\left(1+\frac{\left(4-b^2\right)\sqrt{b^2+8}-b^3}{16-6b^2}\right)$.

Proof. There are four cases to consider:

Case 1 ($x - a > 0$ and $\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)} \geq \frac{2b}{2-b^2}$): In this case, the critical discount factor is determined by manager 2 (see Lemma A.6). By the derivations in Section A.3, the critical discount factor for manager 2 is

$$A_2(\tilde{c}_1, x, a) := \frac{b^2 (1-\tilde{c}_1)^2}{((2-b)(1-\tilde{c}_1) - 2(x-a))^2 - \frac{16}{(4-b^2)^2}((2-b-b^2)(1-\tilde{c}_1) - (2-b^2)(x-a))^2}.$$

Note that $A_2(\tilde{c}_1, x, a) = A_2(\tilde{c}_1, y, 0)$ where y is defined as $y = x - a$. Also, note that

$$\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)} \geq \frac{2b}{2-b^2} = \frac{1-\tilde{c}_1}{1-(\tilde{c}_1+y)} \geq \frac{2b}{2-b^2} \iff y \geq y_2^* = (1-\tilde{c}_1)\left(1 - \frac{2-b^2}{2b}\right).$$

The remainder of the proof, for this case, shows that $\frac{\partial}{\partial \tilde{c}_1} A_2(\tilde{c}_1, y, 0) > 0$ for y such that $y \in [y_2^*, y_U)$ and $y > 0$. $\frac{\partial}{\partial \tilde{c}_1} A_2(\tilde{c}_1, y, 0) > 0$ when

$$\begin{aligned}
&-\left(((2-b)(1-\tilde{c}_1) - 2y)^2 - \frac{16}{(4-b^2)^2}\left(\left(2-b-b^2\right)(1-\tilde{c}_1) - \left(2-b^2\right)y\right)^2\right) b^2 2(1-\tilde{c}_1) \\
&-b^2(1-\tilde{c}_1)^2\left(-2(2-b)((2-b)(1-\tilde{c}_1) - 2y) + \left(2-b-b^2\right)\frac{32}{(4-b^2)^2}\left(\left(2-b-b^2\right)(1-\tilde{c}_1) - \left(2-b^2\right)y\right)\right) = \\
&-b^2 2(1-\tilde{c}_1)\left(((2-b)(1-\tilde{c}_1))^2 - 4y(2-b)(1-\tilde{c}_1) + 4y^2\right) \\
&+b^2 2(1-\tilde{c}_1)\left(\frac{16}{(4-b^2)^2}\left(\left(2-b-b^2\right)^2(1-\tilde{c}_1)^2 - 2y\left(2-b^2\right)(1-\tilde{c}_1)\left(2-b-b^2\right) + \left(2-b^2\right)^2 y^2\right)\right) \\
&-b^2(1-\tilde{c}_1)^2\left(-2(2-b)((2-b)(1-\tilde{c}_1) - 2y) + \left(2-b-b^2\right)\frac{32}{(4-b^2)^2}\left(\left(2-b-b^2\right)(1-\tilde{c}_1) - \left(2-b^2\right)y\right)\right) > 0.
\end{aligned} \tag{6}$$

Dividing the left hand side of (6) by $-2b^2(1-\tilde{c}_1)$ and simplifying yields

$$
\begin{aligned}
&((2-b)(1-\tilde{c}_1))^2 - 4y(2-b)(1-\tilde{c}_1) + 4y^2 \\
&-\frac{16}{(4-b^2)^2}\left(\left(2-b-b^2\right)^2(1-\tilde{c}_1)^2 - 2y\left(2-b^2\right)(1-\tilde{c}_1)\left(2-b-b^2\right) + \left(2-b^2\right)^2 y^2\right) \\
&+(1-\tilde{c}_1)\left(-(2-b)((2-b)(1-\tilde{c}_1)-2y) + \left(2-b-b^2\right)\frac{16}{(4-b^2)^2}\left(\left(2-b-b^2\right)(1-\tilde{c}_1) - \left(2-b^2\right)y\right)\right) = \\
&((2-b)(1-\tilde{c}_1))^2 - 4y(2-b)(1-\tilde{c}_1) + 4y^2 \\
&-\frac{16}{(4-b^2)^2}\left(\left(2-b-b^2\right)^2(1-\tilde{c}_1)^2 - 2y\left(2-b^2\right)(1-\tilde{c}_1)\left(2-b-b^2\right) + \left(2-b^2\right)^2 y^2\right) \\
&-(2-b)\left((2-b)(1-\tilde{c}_1)^2 - 2y(1-\tilde{c}_1)\right) + \left(2-b-b^2\right)\frac{16}{(4-b^2)^2}\left(\left(2-b-b^2\right)(1-\tilde{c}_1)^2 - \left(2-b^2\right)y(1-\tilde{c}_1)\right) = \\
&((2-b)(1-\tilde{c}_1))^2 - 4y(2-b)(1-\tilde{c}_1) + 4y^2 \\
&-\frac{16}{(4-b^2)^2}\left(\left(2-b-b^2\right)^2(1-\tilde{c}_1)^2 - 2y\left(2-b^2\right)(1-\tilde{c}_1)\left(2-b-b^2\right) + \left(2-b^2\right)^2 y^2\right) \\
&-(2-b)^2(1-\tilde{c}_1)^2 + 2y(2-b)(1-\tilde{c}_1) + \left(2-b-b^2\right)\frac{16}{(4-b^2)^2}\left(2-b-b^2\right)(1-\tilde{c}_1)^2 = \\
&-\frac{16}{(4-b^2)^2}\left(2-b^2\right)\left(2-b-b^2\right)y(1-\tilde{c}_1) \\
&-2y(2-b)(1-\tilde{c}_1) + 4y^2 \\
&-\frac{16}{(4-b^2)^2}\left(\left(2-b-b^2\right)^2(1-\tilde{c}_1)^2 - 2y\left(2-b^2\right)(1-\tilde{c}_1)\left(2-b-b^2\right) + \left(2-b^2\right)^2 y^2\right) \\
&+\left(2-b-b^2\right)\frac{16}{(4-b^2)^2}\left(2-b-b^2\right)(1-\tilde{c}_1)^2 - \frac{16}{(4-b^2)^2}\left(2-b^2\right)\left(2-b-b^2\right)y(1-\tilde{c}_1) = \\
&-2y(2-b)(1-\tilde{c}_1) + 4y^2 \\
&-\frac{16}{(4-b^2)^2}\left(\left(2-b-b^2\right)^2(1-\tilde{c}_1)^2 - 2y\left(2-b^2\right)(1-\tilde{c}_1)\left(2-b-b^2\right) + \left(2-b^2\right)^2 y^2\right) \\
&+\frac{16}{(4-b^2)^2}\left(2-b-b^2\right)^2(1-\tilde{c}_1)^2 - \frac{16}{(4-b^2)^2}\left(2-b^2\right)\left(2-b-b^2\right)y(1-\tilde{c}_1) = \\
&-2y(2-b)(1-\tilde{c}_1) + 4y^2 - \frac{16}{(4-b^2)^2}\left(2-b-b^2\right)^2(1-\tilde{c}_1)^2 \\
&+\frac{16}{(4-b^2)^2}2y\left(2-b^2\right)(1-\tilde{c}_1)\left(2-b-b^2\right) - \frac{16}{(4-b^2)^2}\left(2-b^2\right)^2 y^2 \\
&+\frac{16}{(4-b^2)^2}\left(2-b-b^2\right)^2(1-\tilde{c}_1)^2 - \frac{16}{(4-b^2)^2}\left(2-b^2\right)\left(2-b-b^2\right)y(1-\tilde{c}_1) = \\
&-2y(2-b)(1-\tilde{c}_1) + 4y^2 + \frac{16}{(4-b^2)^2}y\left(2-b^2\right)(1-\tilde{c}_1)\left(2-b-b^2\right) - \frac{16}{(4-b^2)^2}\left(2-b^2\right)^2 y^2
\end{aligned}
$$

Thus, $\frac{\partial}{\partial \tilde{c}_1} A_2(\tilde{c}_1, y, 0) > 0$ if

$$\left(4 - \frac{16}{(4-b^2)^2}\left(2-b^2\right)^2\right)y^2 + \left(\frac{16}{(4-b^2)^2}\left(2-b^2\right)(1-\tilde{c}_1)\left(2-b-b^2\right) - 2(2-b)(1-\tilde{c}_1)\right)y < 0.$$

Dividing both sides by y (which is positive) yields

$$F_1(y) = \left(\underbrace{4 - \frac{16}{(4-b^2)^2}\left(2-b^2\right)^2}_{\text{Term 1}} \right) y + \left(\underbrace{\frac{16}{(4-b^2)^2}\left(2-b^2\right)(1-\tilde{c}_1)\left(2-b-b^2\right) - 2(2-b)(1-\tilde{c}_1)}_{\text{Term 2}} \right) < 0.$$

Note that Term 1 is positive by

$$\begin{aligned} 4 - \left(2-b^2\right)^2 \frac{16}{(4-b^2)^2} &> 0 \\ \left(4-b^2\right)^2 &> 4\left(2-b^2\right)^2 \\ 4-b^2 &> 2\left(2-b^2\right) \\ 4-b^2 &> 4-2b^2 \\ b^2 &> 0 \end{aligned}$$

and Term 2 is negative by

$$\begin{aligned} \frac{16}{(4-b^2)^2}\left(2-b^2\right)(1-\tilde{c}_1)\left(2-b-b^2\right) - 2(2-b)(1-\tilde{c}_1) &< 0 \\ \frac{8}{(4-b^2)^2}\left(2-b^2\right)\left(2-b-b^2\right) - (2-b) &< 0 \\ 8\left(2-b^2\right)\left(2-b-b^2\right) &< (2-b)\left(4-b^2\right)^2 \\ 8\left(4-2b-2b^2-2b^2-b^3+b^4\right) &< (2-b)\left(16-8b^2+b^4\right) \\ 32-16b-16b^2-16b^2-8b^3+8b^4 &< 32-16b^2+2b^4-16b+8b^3-b^5 \\ -16b^2-8b^3+8b^4 &< 2b^4+8b^3-b^5 \\ b^5+6b^4 &< 16b^2+16b^3 \\ b^3+6b^2 &< 16+16b. \end{aligned}$$

Thus, $F_1(y)$ is increasing in y. Therefore, it is sufficient to show that $F_1((1-b)(1-\tilde{c}_1)) < 0$ because $y_U < (1-b)(1-\tilde{c}_1)$ (where the inequality follows from Lemma A.1) and $F_1(y)$ is increasing in y.

$$
\begin{aligned}
F_1((1-b)(1-\tilde{c}_1)) &< 0 \\
(1-\tilde{c}_1)(1-b)\left(4-\frac{16}{(4-b^2)^2}\left(2-b^2\right)^2\right)+(1-\tilde{c}_1)\left(\frac{16}{(4-b^2)^2}\left(2-b^2\right)\left(2-b-b^2\right)-2(2-b)\right) &< 0 \\
(1-b)\left(4-\frac{16}{(4-b^2)^2}\left(2-b^2\right)^2\right)+\left(\frac{16}{(4-b^2)^2}\left(2-b^2\right)\left(2-b-b^2\right)-2(2-b)\right) &< 0 \\
\left(4-\frac{16}{(4-b^2)^2}\left(2-b^2\right)^2+\frac{16}{(4-b^2)^2}\left(2-b^2\right)\left(2-b-b^2\right)-2(2-b)\right)-b\left(4-\frac{16}{(4-b^2)^2}\left(2-b^2\right)^2\right) &< 0 \\
\left(4-b\frac{16}{(4-b^2)^2}\left(2-b^2\right)-2(2-b)\right)-b\left(4-\frac{16}{(4-b^2)^2}\left(2-b^2\right)^2\right) &< 0 \\
\left(-b\frac{16}{(4-b^2)^2}\left(2-b^2\right)+2b\right)-b\left(4-\frac{16}{(4-b^2)^2}\left(2-b^2\right)^2\right) &< 0 \\
-2b-b\frac{16}{(4-b^2)^2}\left(2-b^2\right)+\frac{16}{(4-b^2)^2}\left(2-b^2\right)^2 b &< 0 \\
-1-\frac{8}{(4-b^2)^2}\left(2-b^2\right)+\frac{8}{(4-b^2)^2}\left(2-b^2\right)^2 &< 0 \\
-1+\frac{8}{(4-b^2)^2}\left(2-b^2\right)\left(1-b^2\right) &< 0 \\
8\left(2-b^2\right)\left(1-b^2\right) &< \left(4-b^2\right)^2 \\
8\left(2-2b^2-b^2+b^4\right) &< 16-8b^2+b^4 \\
8\left(2-2b^2-b^2+b^4\right) &< 16-8b^2+b^4 \\
16-16b^2-8b^2+8b^4 &< 16-8b^2+b^4 \\
-16b^2+7b^4 &< 0 \\
7b^2 &< 16
\end{aligned}
$$

which holds as $b<1$.

Case 2 ($x-a>0$ and $\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)}<\frac{2b}{2-b^2}$): . In this case, the critical discount factor is determined by manager 2 (see Lemma A.6). By Section A.3, the critical discount factor is

$$
B_2(\tilde{c}_1,x,a)=\frac{\left(\frac{1-\tilde{c}_1}{2b}\right)\left(\frac{2b+2ab+\tilde{c}_1-1}{2b}-\tilde{c}_2\right)-\frac{1}{1-b^2}\left(\frac{1-\tilde{c}_2+a}{2}\right)\left(\frac{1-\tilde{c}_2+a-b(1-\tilde{c}_1)}{2}\right)}{\left(\frac{1-\tilde{c}_1}{2b}\right)\left(\frac{2b+2ab+\tilde{c}_1-1}{2b}-\tilde{c}_2\right)-\frac{((2-b^2)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{(4-b^2)^2(1-b^2)}}.
$$

Note that $B_2(\tilde{c}_1,x,a)=B_2(\tilde{c}_1,y,0)$ where y is defined as $y:=x-a$. Also, note that

$$
\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)}<\frac{2b}{2-b^2}=\frac{1-\tilde{c}_1}{1-(\tilde{c}_1+y)}<\frac{2b}{2-b^2}\iff y<y_2^*=(1-\tilde{c}_1)\left(1-\frac{2-b^2}{2b}\right).
$$

The remainder of the proof for this case, shows that $\frac{\partial}{\partial\tilde{c}_1}B_2(\tilde{c}_1,y,0)>0$ for y such that $y\in(0,\min\{y_2^*,y_U\})$. Routine computations show that $\frac{\partial}{\partial\tilde{c}_1}B_2(\tilde{c}_1,y,0)>0$ when

$$
M_1y^2+M_2y+M_3>0
$$

where

$$
\begin{aligned}
M_1 &= 2b^5 - 14b^3 + 16b \\
M_2 &= 32b\tilde{c}_1 - 16\tilde{c}_1 - 32b + 14b^2\tilde{c}_1 - 28b^3\tilde{c}_1 \\
&\quad - 6b^4\tilde{c}_1 + 4b^5\tilde{c}_1 - 14b^2 + 28b^3 + 6b^4 - 4b^5 + 16 \\
M_3 &= b^5\tilde{c}_1^2 - 2b^5\tilde{c}_1 + b^5 - 6b^4\tilde{c}_1^2 + 12b^4\tilde{c}_1 - 6b^4 - 17b^3\tilde{c}_1^2 + 34b^3\tilde{c}_1 - 17b^3 \\
&\quad + 14b^2\tilde{c}_1^2 - 28b^2\tilde{c}_1 + 14b^2 + 24b\tilde{c}_1^2 - 48b\tilde{c}_1 + 24b - 16\tilde{c}_1^2 + 32\tilde{c}_1 - 16.
\end{aligned}
$$

$M_1y^2 + M_2y + M_3$ is an upward facing parabola with roots (by the quadratic formula) given by

$$
\begin{aligned}
r &= \frac{(16b + 8\tilde{c}_1 \pm D - 16b\tilde{c}_1 - 7b^2\tilde{c}_1 + 14b^3\tilde{c}_1 + 3b^4\tilde{c}_1 - 2b^5\tilde{c}_1 + 7b^2 - 14b^3 - 3b^4 + 2b^5 - (\pm\tilde{c}_1 D) - 8)}{2(b^5 - 7b^3 + 8b)} \\
&= \frac{(16b\,(1-\tilde{c}_1) - 8\,(1-\tilde{c}_1) \pm D\,(1-\tilde{c}_1) + 7b^2\,(1-\tilde{c}_1) - 14b^3\,(1-\tilde{c}_1) - 3b^4\,(1-\tilde{c}_1) + 2b^5\,(1-\tilde{c}_1))}{2(b^5 - 7b^3 + 8b)} \\
&= (1-\tilde{c}_1)\,\frac{(16b - 8 \pm D + 7b^2 - 14b^3 - 3b^4 + 2b^5)}{2(b^5 - 7b^3 + 8b)} \\
&= (1-\tilde{c}_1)\left(1 + \frac{-8 \pm D + 7b^2 - 3b^4}{2(b^5 - 7b^3 + 8b)}\right)
\end{aligned}
$$

where $D = \sqrt{(b^2+8)(b-1)(b+1)(2b^6 - 13b^4 + 23b^2 - 8)}$. If the roots are complex, $M_1y^2 + M_2y + M_3 > 0$ always holds and the proof is complete. If one or more roots are real, then the proof is complete if the smaller of the two roots is greater than $\min\{y_2^*, y_U\}$.

$$
\begin{aligned}
(1-\tilde{c}_1)\left(1 - \frac{2-b^2}{2b}\right) &< (1-\tilde{c}_1)\left(1 + \frac{-8 - D + 7b^2 - 3b^4}{2(b^5 - 7b^3 + 8b)}\right) \\
1 - \frac{2-b^2}{2b} &< 1 + \frac{-8 - D + 7b^2 - 3b^4}{2(b^5 - 7b^3 + 8b)} \\
-\frac{2-b^2}{2b} &< \frac{-8 - D + 7b^2 - 3b^4}{2\,(b^5 - 7b^3 + 8b)}
\end{aligned}
$$

which holds.

Case 3 ($x - a < 0$ and $\frac{1-(\tilde{c}_2 - a)}{1-\tilde{c}_1} \geq \frac{2b}{2-b^2}$): In this case, the critical discount factor is determined by manager 1 (see Lemma A.6). By Section A.3, the critical discount factor of manager 1 in this case is

$$
A_1(\tilde{c}_1, x, a) = \frac{b^2\,(1 - (\tilde{c}_1 + x) + a)^2}{(b + 2\tilde{c}_1 + ab - b\,(\tilde{c}_1 + x) - 2)^2 - \frac{16(b + 2\tilde{c}_1 + ab - b(\tilde{c}_1 + x) - b^2\tilde{c}_1 + b^2 - 2)^2}{(4-b^2)^2}}.
$$

Note that $A_1(\tilde{c}_1, x, a) = A_1(\tilde{c}_1, y, 0)$ where y is defined as $y = x - a$. Also, note that

$$
\frac{1 - (\tilde{c}_2 - a)}{1 - \tilde{c}_1} = \frac{1 - (\tilde{c}_1 + y)}{1 - \tilde{c}_1} \geq \frac{2b}{2-b^2} \iff y \leq y_1^* = (1-\tilde{c}_1)\left(1 - \frac{2b}{2-b^2}\right).
$$

Thus, the remainder of the proof, for this case, shows that $\frac{\partial}{\partial \tilde{c}_1} A_1(\tilde{c}_1, y, 0) > 0$ for y such that $y \in (y_L, y_1^*]$

and $y < 0$. $\frac{\partial}{\partial \tilde{c}_1} A_1(\tilde{c}_1, y, 0) > 0$ when

$$
\begin{aligned}
&-\left(\left((2-b)(1-\tilde{c}_1)+by\right)^2-\frac{16}{\left(4-b^2\right)^2}\left(\left(2-b-b^2\right)(1-\tilde{c}_1)+by\right)^2\right)2b^2(1-\tilde{c}_1-y) \\
&-b^2\left(1-(\tilde{c}_1+y)\right)^2\left(-2(2-b)\left((2-b)(1-\tilde{c}_1)+by\right)+\left(2-b-b^2\right)\frac{32}{\left(4-b^2\right)^2}\left(\left(2-b-b^2\right)(1-\tilde{c}_1)+by\right)\right)= \\
&-\left(\left((2-b)(1-\tilde{c}_1)+by\right)^2-\frac{16}{\left(4-b^2\right)^2}\left(\left(2-b-b^2\right)(1-\tilde{c}_1)+by\right)^2\right)2b^2(1-\tilde{c}_1-y) \\
&-b^2\left(1-(\tilde{c}_1+y)\right)^2\left(-2(2-b)^2(1-\tilde{c}_1)-2(2-b)by+\left(2-b-b^2\right)\frac{32}{\left(4-b^2\right)^2}\left(\left(2-b-b^2\right)(1-\tilde{c}_1)+by\right)\right)>0.
\end{aligned}
\tag{7}
$$

Dividing the left hand side of Equation (7) by $-2b^2(1-\tilde{c}_1-y)$ (note that this quantity is negative) and

simplifying yields

$$\left(\left(\left(2-b\right)\left(1-\tilde{c}_1\right)+by\right)^2 - \frac{16}{\left(4-b^2\right)^2}\left(\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)+by\right)^2 \right)$$

$$+\left(1-\left(\tilde{c}_1+y\right)\right)\left(-\left(2-b\right)^2\left(1-\tilde{c}_1\right)-\left(2-b\right)by+\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)^2\left(1-\tilde{c}_1\right)+\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)by\right)=$$

$$\left(\left(2-b\right)\left(1-\tilde{c}_1\right)\right)^2+2by\left(2-b\right)\left(1-\tilde{c}_1\right)+b^2y^2-\frac{16}{\left(4-b^2\right)^2}\left(\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)\right)^2$$

$$-\frac{16}{\left(4-b^2\right)^2}2by\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)-\frac{16}{\left(4-b^2\right)^2}b^2y^2$$

$$+\left(1-\left(\tilde{c}_1+y\right)\right)\left(-\left(2-b\right)^2\left(1-\tilde{c}_1\right)-\left(2-b\right)by+\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)^2\left(1-\tilde{c}_1\right)+\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)by\right)=$$

$$\left(\left(2-b\right)\left(1-\tilde{c}_1\right)\right)^2+2by\left(2-b\right)\left(1-\tilde{c}_1\right)+b^2y^2-$$

$$\frac{16}{\left(4-b^2\right)^2}\left(\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)\right)^2-\frac{16}{\left(4-b^2\right)^2}2by\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)-\frac{16}{\left(4-b^2\right)^2}b^2y^2$$

$$+\left(1-\tilde{c}_1\right)\left(-\left(2-b\right)^2\left(1-\tilde{c}_1\right)-\left(2-b\right)by+\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)^2\left(1-\tilde{c}_1\right)+\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)by\right)$$

$$-y\left(-\left(2-b\right)^2\left(1-\tilde{c}_1\right)-\left(2-b\right)by+\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)^2\left(1-\tilde{c}_1\right)+\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)by\right)=$$

$$\left(\left(2-b\right)\left(1-\tilde{c}_1\right)\right)^2+2by\left(2-b\right)\left(1-\tilde{c}_1\right)+b^2y^2$$

$$-\frac{16}{\left(4-b^2\right)^2}\left(\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)\right)^2-\frac{16}{\left(4-b^2\right)^2}2by\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)-\frac{16}{\left(4-b^2\right)^2}b^2y^2$$

$$+\left(-\left(2-b\right)^2\left(1-\tilde{c}_1\right)^2-\left(2-b\right)by\left(1-\tilde{c}_1\right)+\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)^2\left(1-\tilde{c}_1\right)^2+\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)by\left(1-\tilde{c}_1\right)\right)$$

$$-y\left(-\left(2-b\right)^2\left(1-\tilde{c}_1\right)-\left(2-b\right)by+\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)^2\left(1-\tilde{c}_1\right)+\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)by\right)=$$

$$\left(by\left(2-b\right)\left(1-\tilde{c}_1\right)+b^2y^2-\frac{16}{\left(4-b^2\right)^2}by\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)-\frac{16}{\left(4-b^2\right)^2}b^2y^2\right)$$

$$-y\left(-\left(2-b\right)^2\left(1-\tilde{c}_1\right)-\left(2-b\right)by+\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)^2\left(1-\tilde{c}_1\right)+\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)by\right)=$$

$$\left(by\left(2-b\right)\left(1-\tilde{c}_1\right)+b^2y^2-\frac{16}{\left(4-b^2\right)^2}by\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)-\frac{16}{\left(4-b^2\right)^2}b^2y^2\right)$$

$$+y\left(2-b\right)^2\left(1-\tilde{c}_1\right)+\left(2-b\right)by^2-\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)^2\left(1-\tilde{c}_1\right)y-\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)by^2=$$

$$y^2\left(b^2-\frac{16}{\left(4-b^2\right)^2}b^2+\left(2-b\right)b-\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)b\right)$$

$$+y\left(b\left(2-b\right)\left(1-\tilde{c}_1\right)-\frac{16}{\left(4-b^2\right)^2}b\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)+\left(2-b\right)^2\left(1-\tilde{c}_1\right)-\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)^2\left(1-\tilde{c}_1\right)\right)=$$

$$y^2\left(b^2-\frac{16}{\left(4-b^2\right)^2}b^2+\left(2-b\right)b-\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)b\right)$$

$$+y\left(1-\tilde{c}_1\right)\left(b\left(2-b\right)-\frac{16}{\left(4-b^2\right)^2}b\left(2-b-b^2\right)+\left(2-b\right)^2-\frac{16}{\left(4-b^2\right)^2}\left(2-b-b^2\right)^2\right)$$

$$y^2\left(2b-\frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)b\right)+y\left(1-\tilde{c}_1\right)\left(2\left(2-b\right)-\frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)\left(2-b-b^2\right)\right).$$

Dividing both sides by y (which is negative) yields

$$F_2(y)=y\left(\underbrace{2b-\frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)b}_{\text{Term 1}}\right)+\left(1-\tilde{c}_1\right)\left(\underbrace{2\left(2-b\right)-\frac{16}{\left(4-b^2\right)^2}\left(2-b^2\right)\left(2-b-b^2\right)}_{\text{Term 2}}\right)>0.$$

Note that Term 1 is positive by

$$
\begin{aligned}
2b - \frac{16}{(4-b^2)^2}\left(2-b^2\right) b &> 0 \\
2b &> \frac{16}{(4-b^2)^2}\left(2-b^2\right) b \\
1 &> \frac{8}{(4-b^2)^2}\left(2-b^2\right) \\
\left(4-b^2\right)^2 &> 8\left(2-b^2\right) \\
16 - 8b^2 + b^4 &> 16 - 8b^2 \\
b^4 &> 0.
\end{aligned}
$$

Note that Term 2 is positive by

$$
\begin{aligned}
2(2-b) - \frac{16}{(4-b^2)^2}\left(2-b^2\right)\left(2-b-b^2\right) &> 0 \\
2(2-b)\left(4-b^2\right)^2 &> 16\left(2-b^2\right)\left(2-b-b^2\right) \\
(2-b)\left(4-b^2\right)^2 &> 8\left(2-b^2\right)\left(2-b-b^2\right) \\
(2-b)\left(16-8b^2+b^4\right) &> 8\left(4-2b-2b^2-2b^2+b^3+b^4\right) \\
(2-b)\left(16-8b^2+b^4\right) &> 8\left(4-2b-4b^2+b^3+b^4\right) \\
32-16b^2+2b^4-\left(16b-8b^3+b^5\right) &> 32-16b-32b^2+8b^3+8b^4 \\
32-16b^2+2b^4-16b+8b^3-b^5 &> 32-16b-32b^2+8b^3+8b^4 \\
16b^2-b^5 &> 6b^4 \\
16b^2 &> 6b^4+b^5 \\
16 &> b^2+b^3.
\end{aligned}
$$

Thus, $F_2(y)$ is increasing in y. Therefore, it is sufficient to show that $F_2(-(1-\tilde{c}_1)(1-b)) > 0$ because $-(1-\tilde{c}_1)(1-b) < y_L$ (see Lemma A.1) and $F_2(y)$ is increasing in y.

$$
\begin{aligned}
F_2(y) &> 0 \\
-(1-\tilde{c}_1)(1-b)\left(2b - \frac{16}{(4-b^2)^2}\left(2-b^2\right)b\right) + (1-\tilde{c}_1)\left(2(2-b) - \frac{16}{(4-b^2)^2}\left(2-b^2\right)\left(2-b-b^2\right)\right) &> 0 \\
-(1-b)\left(2b - \frac{16}{(4-b^2)^2}\left(2-b^2\right)b\right) + \left(2(2-b) - \frac{16}{(4-b^2)^2}\left(2-b^2\right)\left(2-b-b^2\right)\right) &> 0 \\
-\left(2b-2b^2 - \frac{16}{(4-b^2)^2}\left(2-b^2\right)b + b^2\frac{16}{(4-b^2)^2}\left(2-b^2\right)\right) + \left(4-2b - \frac{16}{(4-b^2)^2}\left(2-b^2\right)\left(2-b-b^2\right)\right) &> 0 \\
-2b+2b^2+\frac{16}{(4-b^2)^2}\left(2-b^2\right)b - b^2\frac{16}{(4-b^2)^2}\left(2-b^2\right) + 4 - 2b - \frac{16}{(4-b^2)^2}\left(2-b^2\right)\left(2-b-b^2\right) &> 0 \\
4-2b+2b^2+\frac{16}{(4-b^2)^2}\left(2-b^2\right)\left(b-b^2-\left(2-b-b^2\right)\right) &> 0 \\
4-4b+2b^2+\frac{16}{(4-b^2)^2}\left(2-b^2\right)(2b-2) &> 0 \\
2-2b+b^2-\frac{16}{(4-b^2)^2}\left(2-b^2\right)(1-b) &> 0 \\
2-2b+b^2 &> \frac{16}{(4-b^2)^2}\left(2-b^2\right)(1-b) \\
\left(4-b^2\right)^2\left(2-2b+b^2\right) &> 16\left(2-b^2\right)(1-b) \\
b^4+16 &> 2b^2(b+3)
\end{aligned}
$$

which holds for all b.

Case 4 ($x - a < 0$ and $\frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} < \frac{2b}{2-b^2}$): In this case, the critical discount factor is determined by manager 1 (see Lemma A.6). By Section A.3, the critical discount factor is

$$B_1(\tilde{c}_1, x, a) = \frac{\frac{1-\tilde{c}_2+a}{2b}\left(\frac{2b-1+\tilde{c}_2-a}{2b} - \tilde{c}_1\right) - \frac{1}{1-b^2}\left(\frac{1-\tilde{c}_1}{2}\right)\left(\frac{1-\tilde{c}_1-b(1-\tilde{c}_2+a)}{2}\right)}{\frac{1-\tilde{c}_2+a}{2b}\left(\frac{2b-1+\tilde{c}_2-a}{2b} - \tilde{c}_1\right) - \frac{((2-b^2)(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{(4-b^2)^2(1-b^2)}}$$

Note that $B_1(\tilde{c}_1, x, a) = B_1(\tilde{c}_1, y, 0)$ where y is defined as $y = x - a$. Also, note that

$$\frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1} = \frac{1-(\tilde{c}_1+y)}{1-\tilde{c}_1} < \frac{2b}{2-b^2} \iff y > y_1^* = (1-\tilde{c}_1)\left(1 - \frac{2b}{2-b^2}\right).$$

Thus, the remainder of the proof, for this case, shows that $\frac{\partial}{\partial \tilde{c}_1} B_1(\tilde{c}_1, y, 0) > 0$ for y such that $y \in (\max\{y_L, y_1^*\}, 0)$. Routine computations show that $\frac{\partial}{\partial \tilde{c}_1} B_1(\tilde{c}_1, y, 0) > 0$ when

$$M_1 y^2 + M_2 y + M_3 > 0$$

where

$$\begin{aligned}
M_1 &= -b^5 - 3b^3 + 8b \\
M_2 &= 16b\tilde{c}_1 - 16\tilde{c}_1 - 16b + 14b^2\tilde{c}_1 - 6b^3\tilde{c}_1 - 6b^4\tilde{c}_1 \\
&\quad - 2b^5\tilde{c}_1 - 14b^2 + 6b^3 + 6b^4 + 2b^5 + 16 \\
M_3 &= b^5\tilde{c}_1^2 - 2b^5\tilde{c}_1 + b^5 - 6b^4\tilde{c}_1^2 + 12b^4\tilde{c}_1 - 6b^4 - 17b^3\tilde{c}_1^2 + 34b^3\tilde{c}_1 - 17b^3 \\
&\quad + 14b^2\tilde{c}_1^2 - 28b^2\tilde{c}_1 + 14b^2 + 24b\tilde{c}_1^2 - 48b\tilde{c}_1 + 24b - 16\tilde{c}_1^2 + 32\tilde{c}_1 - 16.
\end{aligned}$$

$M_1 y^2 + M_2 y + M_3$ is an upward facing parabola with roots (by the quadratic formula) given by

$$\begin{aligned}
r &= \frac{8b + 8\tilde{c}_1 \pm D - 8b\tilde{c}_1 - 7b^2\tilde{c}_1 + 3b^3\tilde{c}_1 + 3b^4\tilde{c}_1 + b^5\tilde{c}_1 + 7b^2 - 3b^3 - 3b^4 - b^5 - (\pm\tilde{c}_1 D) - 8}{8b - b^5 - 3b^3} \\
&= \frac{8b(1-\tilde{c}_1) + 8(\tilde{c}_1 - 1) \pm D(1-\tilde{c}_1) - 3b^3(1-\tilde{c}_1) - 3b^4(1-\tilde{c}_1) + b^5(\tilde{c}_1 - 1) + 7b^2(1-\tilde{c}_1)}{8b - b^5 - 3b^3} \\
&= (1-\tilde{c}_1)\frac{8b - 8 \pm D - 3b^3 - 3b^4 - b^5 + 7b^2}{8b - b^5 - 3b^3} \\
&= (1-\tilde{c}_1)\left(1 + \frac{-8 \pm D - 3b^4 + 7b^2}{8b - 3b^3 - b^5}\right) \\
&= (1-\tilde{c}_1)\left(1 - \frac{8 \pm D + 3b^4 - 7b^2}{8b - 3b^3 - b^5}\right)
\end{aligned}$$

where $D = \sqrt{(b^2+8)(b-1)(b+1)(2b^6 - 13b^4 + 23b^2 - 8)}$. If the roots are complex, $M_1 y^2 + M_2 y + M_3 > 0$ always holds and the proof is complete. If one or more roots are real, then the proof is complete if the larger of the two roots is less than y_1^*.

$$\begin{aligned}
(1-\tilde{c}_1)\left(1 - \frac{8 - D + 3b^4 - 7b^2}{8b - 3b^3 - b^5}\right) &< (1-\tilde{c}_1)\left(1 - \frac{2b}{2-b^2}\right) \\
1 - \frac{8 - D + 3b^4 - 7b^2}{8b - 3b^3 - b^5} &< 1 - \frac{2b}{2-b^2} \\
\frac{2b}{2-b^2} &< \frac{8 - D + 3b^4 - 7b^2}{8b - 3b^3 - b^5}
\end{aligned}$$

which holds. □

A.5.3 Proof of Lemma 2: Additional Details

Lemma (Lemma 2 from the Text Appendix). *Suppose $x-a\neq 0$. Then, $\delta^*(\tilde{c}_1,x,a)$ is increasing in x when $x>a$ and decreasing in x when $x<a$.*

Proof. There are four cases to consider:

Case 1 ($x-a>0$ and $\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)}\geq\frac{2b}{2-b^2}$): By the derivations in Section A.3, the critical discount factor is

$$A_2(\tilde{c}_1,x,a)=\frac{b^2\left(1-\tilde{c}_1\right)^2}{\left(\left(2-b\right)\left(1-\tilde{c}_1\right)-2\left(x-a\right)\right)^2-\frac{16}{(4-b^2)^2}\left(\left(2-b-b^2\right)\left(1-\tilde{c}_1\right)-\left(2-b^2\right)\left(x-a\right)\right)^2}.$$

By Lemma A.2, $A_2(\tilde{c}_1,x,a)$ is increasing in x when $\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)}\geq\frac{2b}{2-b^2}$.

Case 2 ($x-a>0$ and $\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)}<\frac{2b}{2-b^2}$): . By the derivations of Section A.3, the critical discount factor is

$$B_2(\tilde{c}_1,x,a)=\frac{\left(\frac{1-\tilde{c}_1}{2b}\right)\left(\frac{2b+2ab+\tilde{c}_1-1}{2b}-\tilde{c}_2\right)-\frac{1}{1-b^2}\left(\frac{1-\tilde{c}_2+a}{2}\right)\left(\frac{1-\tilde{c}_2+a-b(1-\tilde{c}_1)}{2}\right)}{\left(\frac{1-\tilde{c}_1}{2b}\right)\left(\frac{2b+2ab+\tilde{c}_1-1}{2b}-\tilde{c}_2\right)-\frac{((2-b^2)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{(4-b^2)^2(1-b^2)}}.$$

By Lemma A.3, $B_2(\tilde{c}_1,x,a)$ is increasing in x when $\frac{1-\tilde{c}_1}{1-(\tilde{c}_2-a)}<\frac{2b}{2-b^2}$.

Case 3 ($x-a<0$ and $\frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1}\geq\frac{2b}{2-b^2}$): By the derivations of Section A.3, the critical discount factor is

$$A_1(\tilde{c}_1,x,a)=\frac{b^2\left(1-\left(\tilde{c}_1+x\right)+a\right)^2}{\left(b+2\tilde{c}_1+ab-b\left(\tilde{c}_1+x\right)-2\right)^2-\frac{16(b+2\tilde{c}_1+ab-b(\tilde{c}_1+x)-b^2\tilde{c}_1+b^2-2)^2}{(4-b^2)^2}}.$$

By Lemma A.4, $A_1(\tilde{c}_1,x,a)$ is decreasing in x when $\frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1}\geq\frac{2b}{2-b^2}$.

Case 4 ($x-a<0$ and $\frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1}<\frac{2b}{2-b^2}$): By the derivations of Section A.3, the critical discount factor is

$$B_1(\tilde{c}_1,x,a)=\frac{\frac{1-\tilde{c}_2+a}{2b}\left(\frac{2b-1+\tilde{c}_2-a}{2b}-\tilde{c}_1\right)-\frac{1}{1-b^2}\left(\frac{1-\tilde{c}_1}{2}\right)\left(\frac{1-\tilde{c}_1-b(1-\tilde{c}_2+a)}{2}\right)}{\frac{1-\tilde{c}_2+a}{2b}\left(\frac{2b-1+\tilde{c}_2-a}{2b}-\tilde{c}_1\right)-\frac{((2-b^2)(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{(4-b^2)^2(1-b^2)}}$$

By Lemma A.5, $B_1(\tilde{c}_1,x,a)$ is decreasing in x when $\frac{1-(\tilde{c}_2-a)}{1-\tilde{c}_1}<\frac{2b}{2-b^2}$. □

A.5.4 Proof of Proposition 2: Additional Details

Proposition (Proposition 2 from the Main Text). *Suppose $c_1=c_2$ and $a=0$. Then, (i) $\delta^*(\theta_1,\theta_2)=\delta^*(0,0)$ if $\theta_1=\theta_2$, and (ii) $\delta^*(\theta_1,\theta_2)>\delta^*(0,0)$ if $\theta_1\neq\theta_2$.*

Proof. Part i) $\theta_1=\theta_2$, $c_1=c_2$, and $a=0$ imply that $x-a=0$. Substituting $x-a=0$ into $\delta^*(\theta_1,\theta_2)$ (see Subsection (A.3)) yields

$$\delta^*_{sym}=\begin{cases}\frac{(2-b)^2}{b^2-8b+8} & \text{if } \frac{2b}{2-b^2}\leq 1\\ \frac{(2-b)^2\left(1-b-b^2\right)}{4-8b+b^2+3b^3-2b^4} & \text{if } \frac{2b}{2-b^2}>1\end{cases}.$$

Additionally, substituting $c_1=c_2$, $a=0$ and $\theta_1=\theta_2=0$ into $\delta^*(\theta_1,\theta_2)$ yields $\delta(0,0)=\delta^*_{sym}$. Therefore, $\delta^*(\theta_1,\theta_2)=\delta^*(0,0)$ when $\theta_1=\theta_2$.

Part ii) $\theta_1\neq\theta_2$, $c_1=c_2$ and $a=0$ imply that $x>a$ or $x<a$. First, consider the case of $x>a$. By Lemma 2, $\delta^*(\theta_1,\theta_2)$ is increasing in x for all $x>a$ and, thus, $\delta^*(\theta_1,\theta_2)>\delta^*_{sym}$ when $x>a$. Next, consider the case of $x<a$. By Lemma 2, $\delta^*(\theta_1,\theta_2)$ is decreasing in x for all $x<a$ and, thus, $\delta^*(\theta_1,\theta_2)>\delta^*_{sym}$ when $x<a$. As $\delta^*(0,0)=\delta^*_{sym}$ (see part i), it follows that $\delta^*(\theta_1,\theta_2)>\delta^*(0,0)$ if $\theta_1\neq\theta_2$. □

A.5.5 Proof of Proposition 4: Additional Details

Proposition (Proposition 4 from the Main Text). $\pi_1(\theta_1,\theta_2)>\pi_1(0,0)$ *and* $\pi_2(\theta_1,\theta_2)>\pi_2(0,0)$ *if Condition 2 holds.*

Proof. Condition 2b ensures that collusion is sustainable under sales-based compensation and unsustainable under profit-based compensation. Thus, $\pi_i(\theta_1,\theta_2) = \pi_i^C(\theta_1,\theta_2)$ and $\pi_i(0,0) = \pi_i^N(0,0)$ for $i = 1,2$. It remains to show that $\pi_1^C(\theta_1,\theta_2) > \pi_1^N(0,0)$ and $\pi_2^C(\theta_1,\theta_2) > \pi_2^N(0,0)$ for θ_1 and θ_2 satisfying Condition 2.

$\delta^*(0,0) < 1$ ensures both firms are active in the Nash equilibrium (see Subsection A.2.1 and Subsection A.4) under profit-based compensation. When managers collude and sales-based compensation weights are θ_1 and θ_2, $p_1^C = \frac{1+\tilde{c}_1}{2}$ and $p_2^C = \frac{1+\tilde{c}_2+a}{2}$. Thus, firm profits when managers collude and compensation is sales-based are

$$\begin{aligned}\pi_1^C(\theta_1,\theta_2) &= D_1\left(p_1^C,p_2^C\right)\left(p_1^C - c_1\right)\\ &= \frac{1}{2\left(1-b^2\right)}\left(1-\tilde{c}_1-b-ab+b\tilde{c}_2\right)\left(\frac{1+\tilde{c}_1}{2}-c_1\right)\end{aligned}$$

and

$$\begin{aligned}\pi_2^C(\theta_1,\theta_2) &= D_2\left(p_1^C,p_2^C\right)\left(p_2^C - c_1\right)\\ &= \frac{1}{2\left(1-b^2\right)}\left(1-\tilde{c}_2-b+a+b\tilde{c}_1\right)\left(\frac{1+\tilde{c}_2+a}{2}-c_2\right).\end{aligned}$$

Nash equilibrium prices when managers compete and compensation is solely profit-based (i.e., $\theta_1 = \theta_2 = 0$) are

$$p_1^N = \frac{2-b+2c_1-ab+bc_2-b^2}{4-b^2} \tag{8}$$

and

$$p_2^N = \frac{2-b+2c_2+2a+bc_1-ab^2-b^2}{4-b^2}. \tag{9}$$

Thus, firm profits when managers compete and compensation is profit-based are

$$\begin{aligned}\pi_1^N(0,0) &= D_1\left(p_1^N,p_2^N\right)\left(p_1^N - c_1\right)\\ &= D_1\left(p_1^N,p_2^N\right)\left(p_1^N - c_1\right)\\ &= \frac{\left(\left(2-b^2\right)\left(1-c_1\right)-b\left(1-c_2+a\right)\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}\end{aligned}$$

and

$$\begin{aligned}\pi_2^N(0,0) &= D_2\left(p_1^N,p_2^N\right)\left(p_2^N - c_2\right)\\ &= D_2\left(p_1^N,p_2^N\right)\left(p_2^N - c_2\right)\\ &= \frac{\left(\left(2-b^2\right)\left(1-c_2+a\right)-b\left(1-c_1\right)\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}.\end{aligned}$$

For firm 1, it follows that

$$\begin{aligned}&\pi_1^C(\theta_1,\theta_2) > \pi_1^N(0,0)\\ \iff \frac{1}{2\left(1-b^2\right)}\left(\frac{1+\tilde{c}_1}{2}-c_1\right)\left(1-\tilde{c}_1-b-ab+b\tilde{c}_2\right) &> \frac{\left(\left(2-b^2\right)\left(1-c_1\right)-b\left(1-c_2+a\right)\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}\\ \iff \frac{1}{2}\left(\frac{1+\tilde{c}_1}{2}-c_1\right)\left(1-\tilde{c}_1-b-ab+b\tilde{c}_2\right) &> \frac{\left(\left(2-b^2\right)\left(1-c_1\right)-b\left(1-c_2+a\right)\right)^2}{\left(4-b^2\right)^2}\end{aligned}$$

which holds if (note that $1-c_1>\theta_1 c_1$ (which is assumed in Condition 2a) $\implies 1-2c_1+\tilde{c}_1>0$)

$$\begin{aligned}
&\iff \tilde{c}_2 > \frac{b+\tilde{c}_1+ab-1}{b}+4\frac{\left(b\left(1-c_2+a\right)-\left(2-b^2\right)\left(1-c_1\right)\right)^2}{b\left(4-b^2\right)^2\left(1-2c_1+\tilde{c}_1\right)}\\
&\iff \tilde{c}_2 > \frac{b+\tilde{c}_1+ab-1}{b}+4\frac{\left(b\left(1-c_2+a\right)-\left(2-b^2\right)\left(1-c_1\right)\right)^2}{b\left(4-b^2\right)^2\left(1-2c_1+\tilde{c}_1\right)}\\
&\iff c_2\left(1-\theta_2\right) > \frac{b+\tilde{c}_1+ab-1}{b}+4\frac{\left(b\left(1-c_2+a\right)-\left(2-b^2\right)\left(1-c_1\right)\right)^2}{b\left(4-b^2\right)^2\left(1-2c_1+\tilde{c}_1\right)}\\
&\iff -c_2\theta_2 > \frac{b+\tilde{c}_1+ab-1-bc_2}{b}+4\frac{\left(b\left(1-c_2+a\right)-\left(2-b^2\right)\left(1-c_1\right)\right)^2}{b\left(4-b^2\right)^2\left(1-2c_1+\tilde{c}_1\right)}\\
&\iff -c_2\theta_2 > \frac{b\left(1-c_2+a\right)-\left(1-\tilde{c}_1\right)}{b}+4\frac{\left(b\left(1-c_2+a\right)-\left(2-b^2\right)\left(1-c_1\right)\right)^2}{b\left(4-b^2\right)^2\left(1-2c_1+\tilde{c}_1\right)}\\
&\iff \theta_2 < \frac{\left(1-\tilde{c}_1\right)-b\left(1-c_2+a\right)}{bc_2}+4\frac{\left(\left(2-b^2\right)\left(1-c_1\right)-b\left(1-c_2+a\right)\right)^2}{b\left(4-b^2\right)^2\left(2c_1-\tilde{c}_1-1\right)c_2} \qquad (10)
\end{aligned}$$

which holds by Condition 2a. For firm 2, it follows that

$$\begin{aligned}
&\pi_2^C(\theta_1,\theta_2) > \pi_2^N(0,0)\\
\iff &\frac{1}{2\left(1-b^2\right)}\left(1-\tilde{c}_2-b+a+b\tilde{c}_1\right)\left(\frac{1+\tilde{c}_2+a}{2}-c_2\right) > \frac{\left(\left(2-b^2\right)\left(1-c_2+a\right)-b\left(1-c_1\right)\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}\\
\iff &\frac{1}{2}\left(1-\tilde{c}_2-b+a+b\tilde{c}_1\right)\left(\frac{1+\tilde{c}_2+a}{2}-c_2\right) > \frac{\left(\left(2-b^2\right)\left(1-c_2+a\right)-b\left(1-c_1\right)\right)^2}{\left(4-b^2\right)^2}
\end{aligned}$$

which holds if (note that $1-c_2+a>\theta_2 c_2$ (which is assumed in Condition 2a) $\implies 1+\tilde{c}_2-2c_2+a>0$)

$$\begin{aligned}
&\iff \tilde{c}_1 > \frac{b+\tilde{c}_2-a-1}{b}+4\frac{\left(\left(2-b^2\right)\left(1-c_2+a\right)-b\left(1-c_1\right)\right)^2}{b\left(4-b^2\right)^2\left(1-2c_2+\tilde{c}_2+a\right)}\\
&\iff \tilde{c}_1 > \frac{b+\tilde{c}_2-a-1}{b}+4\frac{\left(\left(2-b^2\right)\left(1-c_2+a\right)-b\left(1-c_1\right)\right)^2}{b\left(4-b^2\right)^2\left(1-2c_2+\tilde{c}_2+a\right)}\\
&\iff c_1\left(1-\theta_1\right) > \frac{b+\tilde{c}_2-a-1}{b}+4\frac{\left(\left(2-b^2\right)\left(1-c_2+a\right)-b\left(1-c_1\right)\right)^2}{b\left(4-b^2\right)^2\left(1-2c_2+\tilde{c}_2+a\right)}\\
&\iff -c_1\theta_1 > \frac{b+\tilde{c}_2-a-1-bc_1}{b}+4\frac{\left(\left(2-b^2\right)\left(1-c_2+a\right)-b\left(1-c_1\right)\right)^2}{b\left(4-b^2\right)^2\left(1-2c_2+\tilde{c}_2+a\right)}\\
&\iff -c_1\theta_1 > \frac{b\left(1-c_1\right)-\left(1-\tilde{c}_2+a\right)}{b}+4\frac{\left(\left(2-b^2\right)\left(1-c_2+a\right)-b\left(1-c_1\right)\right)^2}{b\left(4-b^2\right)^2\left(1-2c_2+\tilde{c}_2+a\right)}\\
&\iff \theta_1 < \frac{\left(1-\tilde{c}_2+a\right)-b\left(1-c_1\right)}{bc_1}+4\frac{\left(\left(2-b^2\right)\left(1-c_2+a\right)-b\left(1-c_1\right)\right)^2}{b\left(4-b^2\right)^2\left(2c_2-\tilde{c}_2-a-1\right)c_1} \qquad (11)
\end{aligned}$$

which holds by Condition 2a. □

A.5.6 Proof of Proposition 5: Additional Details

Let $D_i^C(\theta_1,\theta_2)$ denote product i's demand when managers collude and sales weights are θ_1 and θ_2. Let $D_i^N(\theta_1,\theta_2)$ denote product i's demand when managers compete and sales weights are θ_1 and θ_2. Prices under collusion are denoted $p_1^C(\theta_1,\theta_2)$ and $p_2^C(\theta_1,\theta_2)$ while prices under competition are denoted $p_1^N(\theta_1,\theta_2)$

and $p_2^N(\theta_1,\theta_2)$.

$$CS^j(\theta_1,\theta_2) = D_1^j(\theta_1,\theta_2) + (1+a)\,D_2^j(\theta_1,\theta_2) - \frac{1}{2}\left(D_1^j(\theta_1,\theta_2)^2 + D_2^j(\theta_1,\theta_2)^2 + 2bD_1^j(\theta_1,\theta_2)D_2^j(\theta_1,\theta_2)\right)$$
$$- p_1^j(\theta_1,\theta_2)D_1^j(\theta_1,\theta_2) - p_2^j(\theta_1,\theta)D_2^j(\theta_1,\theta_2) + I$$

denotes consumer surplus when managers collude ($j = C$) or compete ($j = N$) and sales weights are θ_1 and θ_2. $I > 0$ denotes income.

Proposition (Proposition 5 from the Main Text). *$CS(\theta_1,\theta_2) < CS(0,0)$ if Condition 3 holds.*

Proof. Condition 3b ensures that collusion is sustainable under sales-based compensation and unsustainable under profit-based compensation. Thus, $CS(\theta_1,\theta_2) = CS^C(\theta_1,\theta_2)$ and $CS(0,0) = CS^N(0,0)$ for $i = 1,2$. It remains to show that $CS^C(\theta_1,\theta_2) < CS^N(0,0)$ for θ_1 and θ_2 satisfying Condition 3.

$\delta^*(0,0) < 1$ ensures both firms are active in the Nash equilibrium (see subsection A.2.1 and subsection A.4) under profit-based compensation. Note that $CS^C(\theta_1,\theta_2) < CS^N(0,0)$ if

$$\begin{aligned}
p_1^N &< p_1^C \\
\frac{2-b+2c_1-ab+bc_2-b^2}{4-b^2} &< \frac{1+\tilde{c}_1}{2} \\
\frac{2-b+2c_1-ab+bc_2-b^2}{4-b^2} &< \frac{1+(1-\theta_1)\,c_1}{2} \\
\theta_1 &< \frac{1}{c_1}\left(1+c_1-2\frac{2-b+2c_1-ab+bc_2-b^2}{4-b^2}\right) \\
\theta_1 &< \frac{1}{c_1}\left(1+c_1-\frac{4-2b+4c_1-2ab+2bc_2-2b^2}{4-b^2}\right) \\
\theta_1 &< \frac{1}{c_1}\left(\frac{4-b^2+4c_1-c_1b^2-4+2b-4c_1+2ab-2bc_2+2b^2}{4-b^2}\right) \\
\theta_1 &< \frac{1}{c_1}\left(\frac{2b-c_1b^2+2ab-2bc_2+b^2}{4-b^2}\right) \\
\theta_1 &< \frac{1}{c_1}\left(\frac{2b\,(1-c_2+a)+b^2\,(1-c_1)}{4-b^2}\right)
\end{aligned}$$

and

$$\begin{aligned}
p_2^N &< p_2^C \\
\frac{2-b+2c_2+2a+bc_1-ab^2-b^2}{4-b^2} &< \frac{1+\tilde{c}_2+a}{2} \\
\frac{2-b+2c_2+2a+bc_1-ab^2-b^2}{4-b^2} &< \frac{1+(1-\theta_2)\,c_2+a}{2} \\
\theta_2 &< \frac{1}{c_2}\left(1+c_2+a-2\frac{2-b+2c_2+2a+bc_1-ab^2-b^2}{4-b^2}\right) \\
\theta_2 &< \frac{1}{c_2}\left(1+c_2+a-\frac{4-2b+4c_2+4a+2bc_1-2ab^2-2b^2}{4-b^2}\right) \\
\theta_2 &< \frac{1}{c_2}\left(1+c_2+a+\frac{-4+2b-4c_2-4a-2bc_1+2ab^2+2b^2}{4-b^2}\right) \\
\theta_2 &< \frac{1}{c_2}\left(\frac{4-b^2+4c_2-b^2c_2+4a-b^2a-4+2b-4c_2-4a-2bc_1+2ab^2+2b^2}{4-b^2}\right) \\
\theta_2 &< \frac{1}{c_2}\left(\frac{-b^2c_2+2b-2bc_1+ab^2+b^2}{4-b^2}\right) \\
\theta_2 &< \frac{1}{c_2}\left(\frac{2b\,(1-c_1)+b^2\,(1-c_2+a)}{4-b^2}\right).
\end{aligned}$$

□

A.5.7 Proof of Lemma 3: Additional Details

Lemma (Lemma 3 from the Text Appendix). *$\pi_i^N(\theta_1,\theta_2)$ is non-increasing in θ_i for $i=1,2$.*

Proof. **Firm 1:** Subsection A.7.1 shows that firm 1's profit in the Nash equilibrium (as a function of θ_1 and θ_2) is

$$\pi^N(\theta_1,\theta_2)=\begin{cases}0 & \text{if } (1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right)\geq x-a\\ \pi^N_{1,Both}(\theta_1,\theta_2) & \text{if } (1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right)\geq x-a>(1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right)\\ \pi^N_{1,Limit}(\theta_1,\theta_2) & \text{if } \left(1-\frac{b}{2}\right)(1-\tilde{c}_1)\geq x-a>(1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right)\\ \pi^N_{1,Monop}(\theta_1,\theta_2) & \text{if } x-a>\left(1-\frac{b}{2}\right)(1-\tilde{c}_1)\end{cases}$$

where $\pi^N_{1,Both}(\theta_1,\theta_2)$ denotes profit if both firms are active in the Nash equilibrium, $\pi^N_{1,Limit}(\theta_1,\theta_2)$ denotes profit if only Firm 1 is active and the limit pricing equilibrium occurs Zanchettin (2006), and $\pi^N_{1,Monop}(\theta_1,\theta_2)$ denotes profit if only Firm 1 is active and sets the monopoly price Zanchettin (2006). Derivations of $\pi^N_{1,Both}(\theta_1,\theta_2)$, $\pi^N_{1,Monop}(\theta_1,\theta_2)$ and $\pi^N_{1,Limit}(\theta_1,\theta_2)$ are presented in Subsection A.7.1. Note that $\pi_1^N(\theta_1,\theta_2)$ is continuous in θ_1 by Lemma A.9.

As $\pi^N_{1,Limit}(\theta_1,\theta_2)$ is constant in θ_1, it suffices to show the following two parts:

Part 1: $\pi^N_{1,Both}(\theta_1,\theta_2)$ is non-increasing for $(1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right)\geq x-a>(1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right)$.

Part 2: $\pi^N_{1,Monop}(\theta_1,\theta_2)$ is non-increasing for $x-a>\left(1-\frac{b}{2}\right)(1-\tilde{c}_1)$.

Firm 1 Part 1:

$$\pi^N_{1,Both}(\theta_1,\theta_2)=\frac{1}{1-b^2}\left(\frac{\left(\left(2-b^2\right)(1-\tilde{c}_1)-b\left(1-\tilde{c}_2+a\right)\right)}{4-b^2}\right)\left(\frac{\left(2-b^2\right)(1-\tilde{c}_1)-b\left(1-\tilde{c}_2+a\right)}{4-b^2}+(\tilde{c}_1-c_1)\right)$$

The partial derivative of $\pi^N_{1,Both}(\theta_1,\theta_2)$ with respect to θ_1 is

$$\begin{aligned}\frac{\partial\pi^N_{1,Both}(\theta_1,\theta_2)}{\partial\theta_1}=&\frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)c_1}{4-b^2}\right]\left(\frac{\left(2-b^2\right)(1-\tilde{c}_1)-b\left(1-\tilde{c}_2+a\right)}{4-b^2}+(\tilde{c}_1-c_1)\right)\\&+\frac{1}{1-b^2}\left[\frac{\left(\left(2-b^2\right)(1-\tilde{c}_1)-b\left(1-\tilde{c}_2+a\right)\right)}{4-b^2}\right]\left(\frac{\left(2-b^2\right)c_1}{4-b^2}-c_1\right)\\=&\frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)c_1}{4-b^2}\right]\left(\frac{\left(2-b^2\right)(1-\tilde{c}_1)-b\left(1-\tilde{c}_2+a\right)}{4-b^2}+((1-c_1)-(1-\tilde{c}_1))\right)\\&+\frac{1}{1-b^2}\left[\frac{\left(\left(2-b^2\right)(1-\tilde{c}_1)-b\left(1-\tilde{c}_2+a\right)\right)}{4-b^2}\right]\left(\frac{\left(2-b^2\right)c_1-\left(4-b^2\right)c_1}{4-b^2}\right)\\=&\frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)c_1}{4-b^2}\right]\left(\frac{\left(2-b^2\right)(1-\tilde{c}_1)-b\left(1-\tilde{c}_2+a\right)+(1-c_1)\left(4-b^2\right)-(1-\tilde{c}_1)\left(4-b^2\right)}{4-b^2}\right)\\&+\frac{1}{1-b^2}\left[\frac{\left(\left(2-b^2\right)(1-\tilde{c}_1)-b\left(1-\tilde{c}_2+a\right)\right)}{4-b^2}\right]\left(\frac{-2c_1}{4-b^2}\right)\\=&\frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)c_1}{4-b^2}\right]\left(\frac{-2(1-\tilde{c}_1)-b\left(1-\tilde{c}_2+a\right)+(1-c_1)\left(4-b^2\right)}{4-b^2}\right)\\&+\frac{1}{1-b^2}\left[\frac{\left(\left(2-b^2\right)(1-\tilde{c}_1)-b\left(1-\tilde{c}_2+a\right)\right)}{4-b^2}\right]\left(\frac{-2c_1}{4-b^2}\right).\end{aligned}\tag{12}$$

First, note that $\pi^N_{1,Both}(\theta_1,\theta_2)$ is a concave parabola in θ_1 as

$$\frac{\partial^2\pi^N_{1,Both}(\theta_1,\theta_2)}{\partial\theta_1^2}=\frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)c_1}{4-b^2}\right]\left(\frac{-2c_1}{4-b^2}\right)+\frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)c_1}{4-b^2}\right]\left(\frac{-2c_1}{4-b^2}\right)<0.$$

Recall that both firms are active in the Nash equilibrium if

$$(1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right) > \tilde{c}_2-\tilde{c}_1-a > (1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right)$$

or

$$\frac{\frac{(2-b^2)}{b}(1-\tilde{c}_2+a)-1+c_1}{c_1} > \theta_1 > \frac{\frac{b}{2-b^2}(1-\tilde{c}_2+a)-1+c_1}{c_1}.$$

Thus, it suffices to show that $\frac{\partial \pi^N_{1,Both}(\theta_1,\theta_2)}{\partial \theta_1} \leq 0$ when $\theta_1 = \frac{\frac{b}{2-b^2}(1-\tilde{c}_2+a)-1+c_1}{c_1}$ or

$$b(1-\tilde{c}_2+a) = \left(2-b^2\right)(1-\tilde{c}_1)$$

holds. Substituting the above into (12) yields

$$\begin{aligned}
\frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)c_1}{4-b^2}\right]\left(\frac{-2(1-\tilde{c}_1)-\left(2-b^2\right)(1-\tilde{c}_1)+(1-c_1)\left(4-b^2\right)}{4-b^2}\right) && \leq 0 \\
\iff -2(1-\tilde{c}_1)-\left(2-b^2\right)(1-\tilde{c}_1)+(1-c_1)\left(4-b^2\right) && \leq 0 \\
\iff -\left(4-b^2\right)(1-\tilde{c}_1)+(1-c_1)\left(4-b^2\right) && \leq 0 \\
\iff -\left(4-b^2\right)(c_1-\tilde{c}_1) && \leq 0
\end{aligned}$$

which holds. Additionally, $\frac{\partial \pi^N_{1,Both}(\theta_1,\theta_2)}{\partial \theta_1} < 0$ (strictly) when $\theta_1 > 0$.

Firm 1 Part 2: Profit under monopoly pricing,

$$\pi^N_{1,Monop}(\theta_1,\theta_2) = \frac{(1-c_1)^2-(c_1\theta_1)^2}{4}$$

is declining in θ_1.

Firm 2: Firm 2's profit in the Nash equilibrium (as a function of θ_1 and θ_2) is

$$\pi^N_2(\theta_1,\theta_2) = \begin{cases} 0 & \text{if } (1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right) \geq x-a \\ \pi^N_{2,Both}(\theta_1,\theta_2) & \text{if } (1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right) > x-a \geq (1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right) \\ \pi^N_{2,Limit}(\theta_1,\theta_2) & \text{if } (1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right) > x-a \geq (1-\tilde{c}_1)\left(1-\frac{2}{b}\right) \\ \pi^N_{2,Monop}(\theta_1,\theta_2) & \text{if } x-a < \left(1-\frac{2}{b}\right)(1-\tilde{c}_1) \end{cases}$$

where $\pi^N_{2,Both}(\theta_1,\theta_2)$ denotes profit if both firms are active in the Nash equilibrium, $\pi^N_{2,Limit}(\theta_1,\theta_2)$ denotes profit if only Firm 2 is active and the limit pricing equilibrium occurs Zanchettin (2006), and $\pi^N_{2,Monop}(\theta_1,\theta_2)$ denotes profit if only Firm 2 is active and sets the monopoly price Zanchettin (2006). Derivations of $\pi^N_{2,Both}(\theta_1,\theta_2)$, $\pi^N_{2,Monop}(\theta_1,\theta_2)$ and $\pi^N_{2,Limit}(\theta_1,\theta_2)$ are presented in Subsection A.7.1. Note that $\pi^N_2(\theta_1,\theta_2)$ is continuous in θ_2 by Lemma A.10.

As $\pi^N_{2,Limit}(\theta_1,\theta_2)$ is constant in θ_2, it suffices to show:

Part 1: $\pi^N_{2,Both}(\theta_1,\theta_2)$ is non-increasing in θ_2 for $(1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right) > x-a \geq (1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right)$.

Part 2: $\pi^N_{2,Monop}(\theta_1,\theta_2)$ is non-increasing in θ_2 for if $x-a < \left(1-\frac{2}{b}\right)(1-\tilde{c}_1)$.

Firm 2 Part 1:

$$\pi^N_{2,Both}(\theta_1,\theta_2) = \frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1)}{4-b^2}\right]\left(\frac{\left(2-b^2\right)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1)}{4-b^2}+(\tilde{c}_2-c_2)\right)$$

The partial derivative of $\pi^N_{2,Both}(\theta_1,\theta_2)$ with respect to θ_2 is

$$\begin{aligned}
\frac{\partial \pi^N_{2,Both}(\theta_1,\theta_2)}{\partial \theta_2} =& \frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)c_2}{4-b^2}\right]\left(\frac{\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)}{4-b^2}+\left(\tilde{c}_2-c_2\right)\right)\\
&+\frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)}{4-b^2}\right]\left(\frac{\left(2-b^2\right)c_2}{4-b^2}-c_2\right)\\
=& \frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)c_2}{4-b^2}\right]\left(\frac{\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)}{4-b^2}+\left(\left(1-c_2+a\right)-\left(1-\tilde{c}_2+a\right)\right)\right)\\
&+\frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)}{4-b^2}\right]\left(\frac{\left(2-b^2\right)c_2-\left(4-b^2\right)c_2}{4-b^2}\right)\\
=& \frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)c_2}{4-b^2}\right]\left(\frac{\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)+\left(1-c_2+a\right)\left(4-b^2\right)-\left(1-\tilde{c}_2+a\right)\left(4-b^2\right)}{4-b^2}\right)\\
&+\frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)}{4-b^2}\right]\left(\frac{-2c_2}{4-b^2}\right)\\
=& \frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)c_2}{4-b^2}\right]\left(\frac{-2\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)+\left(1-c_2+a\right)\left(4-b^2\right)}{4-b^2}\right) \qquad (13)\\
&+\frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)}{4-b^2}\right]\left(\frac{-2c_2}{4-b^2}\right)
\end{aligned}$$

First, note that $\pi^N_{2,Both}(\theta_1,\theta_2)$ is a concave parabola in θ_2 as

$$\frac{\partial^2 \pi^N_{2,Both}(\theta_1,\theta_2)}{\partial \theta_2^2} = \frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)c_2}{4-b^2}\right]\left(\frac{-2c_2}{4-b^2}\right)+\frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)c_2}{4-b^2}\right]\left(\frac{-2c_2}{4-b^2}\right)<0.$$

Recall that both firms are active in the Nash equilibrium if

$$\left(1-\tilde{c}_1\right)\left(1-\frac{b}{2-b^2}\right)>\tilde{c}_2-\tilde{c}_1-a>\left(1-\tilde{c}_1\right)\left(1-\frac{2-b^2}{b}\right)$$

or

$$\frac{\left(1-\tilde{c}_1\right)\left(1-\frac{b}{2-b^2}\right)+\tilde{c}_1+a-c_2}{c_2}<\theta_2<\frac{\left(1-\tilde{c}_1\right)\left(1-\frac{2-b^2}{b}\right)+\tilde{c}_1+a-c_2}{c_2}.$$

Thus, it suffices to show that $\frac{\partial \pi^N_{2,Both}(\theta_1,\theta_2)}{\partial \theta_2}\le 0$ when $\theta_2=\frac{\left(1-\tilde{c}_1\right)\left(1-\frac{b}{2-b^2}\right)+\tilde{c}_1+a-c_2}{c_2}$ or

$$\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)=b\left(1-\tilde{c}_1\right).$$

Substituting the above into (13) yields

$$\begin{aligned}
&\frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)c_2}{4-b^2}\right]\left(\frac{-2\left(1-\tilde{c}_2+a\right)-\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)+\left(1-c_2+a\right)\left(4-b^2\right)}{4-b^2}\right) &&\le 0\\
\iff& -2\left(1-\tilde{c}_2+a\right)-\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)+\left(1-c_2+a\right)\left(4-b^2\right) &&\le 0\\
\iff& -\left(4-b^2\right)\left(1-\tilde{c}_2+a\right)+\left(1-c_2+a\right)\left(4-b^2\right) &&\le 0\\
\iff& -\left(4-b^2\right)\left(c_2-\tilde{c}_2\right) &&\le 0
\end{aligned}$$

which holds. Additionally, $\frac{\partial \pi^N_{2,Both}(\theta_1,\theta_2)}{\partial \theta_2}<0$ (strictly) when $\theta_2>0$.

Firm 2 Part 2: Profit under monopoly pricing,

$$\pi^N_{2,Monop}(\theta_1,\theta_2)=\frac{\left(1-c_2+a\right)^2-\left(c_2\theta_2\right)^2}{4}$$

is declining in θ_2. □

A.5.8 Proof of Proposition 6: Additional Details

Proposition (Proposition 6 from the Main Text). (θ_1^*, θ_2^*) *is collusion-compatible if Condition 4 holds.*

Proof. Condition 4b ($\delta^*(\theta_1, \theta_2) < \delta$) ensures no manager wishes to defect from collusion. It remains to show that owners have no incentive to defect in stage 0 (the initial stage).

Owner 1: First, we show owner 1 has no incentive to defect in stage 0. Note that

$$\begin{aligned}
\pi_1^C(\theta_1, \theta_2) &= D_1\left(p_1^C, p_2^C\right)\left(p_1^C - c_1\right) \\
&= \frac{1}{1-b^2}\left(1 - b - ba - \frac{1+\tilde{c}_1}{2} + b\left(\frac{1+\tilde{c}_2+a}{2}\right)\right)\left(\frac{1+\tilde{c}_1}{2} - c_1\right) \\
&= \frac{1}{2\left(1-b^2\right)}\left(2 - 2b - 2ba - (1+\tilde{c}_1) + b\left(1+\tilde{c}_2+a\right)\right)\left(\frac{1+\tilde{c}_1}{2} - c_1\right) \\
&= \frac{1}{2\left(1-b^2\right)}\left(1 - \tilde{c}_1 - b - ab + b\tilde{c}_2\right)\left(\frac{1+\tilde{c}_1}{2} - c_1\right) \\
&= \frac{1}{2\left(1-b^2\right)}\left(1 - \tilde{c}_1 - b\left(1-\tilde{c}_2+a\right)\right)\left(\frac{1+\tilde{c}_1}{2} - c_1\right).
\end{aligned}$$

As $\pi_1^N(\theta_1, \theta_2)$ is non-increasing in θ_1 (see Lemma 3 from the in-text appendix), it suffices to show

$$\pi_1^N(0, \theta_2) \leq \pi_1^C(\theta_1, \theta_2).$$

$\delta^*(\theta_1, \theta_2) < 1$ implies that (by Lemma A.8)

$$(1-\tilde{c}_1)\left(1 - \frac{\left(4-b^2\right)\sqrt{b^2+8} - b^3}{8}\right) < \tilde{c}_2 - \tilde{c}_1 - a < (1-\tilde{c}_1)\left(1 - \frac{\left(4-b^2\right)\sqrt{b^2+8} + b^3}{16-6b^2}\right).$$

The above inequality implies that (by Lemma A.1)

$$\begin{aligned}
\tilde{c}_2 - \tilde{c}_1 - a &< (1-\tilde{c}_1)\left(1 - \frac{b}{2-b^2}\right) \\
\implies \tilde{c}_2 - \tilde{c}_1 - a &< 1 - \tilde{c}_1 - (1-\tilde{c}_1)\frac{b}{2-b^2} \\
\implies \tilde{c}_2 - a &< 1 - (1-\tilde{c}_1)\frac{b}{2-b^2} \\
\implies \tilde{c}_2 - a &< 1 - (1-c_1)\frac{b}{2-b^2} \\
\implies \tilde{c}_2 - c_1 - a &< 1 - c_1 - (1-c_1)\frac{b}{2-b^2} \\
\implies \tilde{c}_2 - c_1 - a &< (1-c_1)\left(1 - \frac{b}{2-b^2}\right).
\end{aligned}$$

Thus, there are three possible cases in the $(0, \theta_2)$-sub-game:

Case 1: $(0, \theta_2) \in S_{Both}$

Case 2: $(0, \theta_2) \in S_{2,Limit}$

Case 3: $(0, \theta_2) \in S_{2,Monop}$

where S_{Both}, $S_{2,Limit}$ and $S_{2,Monop}$ are defined in Section A.7.1.

Owner 1 Case 1: When $(0, \theta_2) \in S_{both}$,

$$\pi_1^N(0, \theta_2) = \pi_{1,Both}^N(0, \theta_2) = \frac{\left(\left(2-b^2\right)(1-c_1) - b\left(1-\tilde{c}_2+a\right)\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}.$$

Owner 1 has no incentive to defect when

$$
\begin{aligned}
\pi^N_{1,Both}\left(0,\theta_2\right) &\leq \pi^C_1(\theta_1,\theta_2) \\
\frac{\left(\left(2-b^2\right)\left(1-c_1\right)-b\left(1-\tilde{c}_2+a\right)\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)} &\leq \frac{1}{2\left(1-b^2\right)}\left(1-\tilde{c}_1-b\left(1-\tilde{c}_2+a\right)\right)\left(\frac{1+\tilde{c}_1}{2}-c_1\right) \\
\frac{\left(\left(2-b^2\right)\left(1-c_1\right)-b\left(1-\tilde{c}_2+a\right)\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)} &\leq \frac{1}{2\left(1-b^2\right)}\left(1-\tilde{c}_1-b\left(1-\tilde{c}_2+a\right)\right)\left(\frac{1+\tilde{c}_1}{2}-\tilde{c}_1\right) \\
&-\left(c_1-\tilde{c}_1\right)\frac{1}{2\left(1-b^2\right)}\left(1-\tilde{c}_1-b\left(1-\tilde{c}_2+a\right)\right) \\
\frac{\left(\left(2-b^2\right)\left(1-\tilde{c}_1\right)-\left(2-b^2\right)\left(c_1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)} &\leq \frac{1}{2\left(1-b^2\right)}\left(1-\tilde{c}_1-b\left(1-\tilde{c}_2+a\right)\right)\left(\frac{1+\tilde{c}_1}{2}-\tilde{c}_1\right) \\
&-\left(c_1-\tilde{c}_1\right)\frac{1}{2\left(1-b^2\right)}\left(1-\tilde{c}_1-b\left(1-\tilde{c}_2+a\right)\right) \\
\frac{\left(\left(2-b^2\right)\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)} &\leq \frac{1}{2\left(1-b^2\right)}\left(1-\tilde{c}_1-b\left(1-\tilde{c}_2+a\right)\right)\left(\frac{1+\tilde{c}_1}{2}-\tilde{c}_1\right) \\
-2\left(2-b^2\right)\left(c_1-\tilde{c}_1\right)\frac{\left(2-b^2\right)\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)}{\left(4-b^2\right)^2\left(1-b^2\right)} &-\left(c_1-\tilde{c}_1\right)\frac{1}{2\left(1-b^2\right)}\left(1-\tilde{c}_1-b\left(1-\tilde{c}_2+a\right)\right) \\
+\frac{\left(2-b^2\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}\left(c_1-\tilde{c}_1\right)^2 & \\
M^N_1(\theta_1,\theta_2)+\frac{\left(2-b^2\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}\left(c_1-\tilde{c}_1\right)^2 &\leq M^C_1(\theta_1,\theta_2) \\
-2\left(2-b^2\right)\left(c_1-\tilde{c}_1\right)\frac{\left(2-b^2\right)\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)}{\left(4-b^2\right)^2\left(1-b^2\right)} &-\left(c_1-\tilde{c}_1\right)\frac{1}{2\left(1-b^2\right)}\left(1-\tilde{c}_1-b\left(1-\tilde{c}_2+a\right)\right) \\
-2\left(c_1-\tilde{c}_1\right)\left(2-b^2\right)\frac{\left(2-b^2\right)\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)}{\left(4-b^2\right)^2\left(1-b^2\right)} &\leq M^C_1(\theta_1,\theta_2)-M^N_1(\theta_1,\theta_2) \\
\left(c_1-\tilde{c}_1\right)\left(\frac{1}{2\left(1-b^2\right)}\left(1-\tilde{c}_1-b\left(1-\tilde{c}_2+a\right)\right)\right) & \\
+\frac{\left(2-b^2\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}\left(c_1-\tilde{c}_1\right)^2 & \\
\left(c_1-\tilde{c}_1\right)\left(D^C_1(\theta_1,\theta_2)-\frac{2\left(2-b^2\right)}{\left(4-b^2\right)}D^N_1(\theta_1,\theta_2)\right) &\leq M^C_1(\theta_1,\theta_2)-M^N_1(\theta_1,\theta_2). \\
+\frac{\left(2-b^2\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}\left(c_1-\tilde{c}_1\right)^2 &
\end{aligned}
$$

Solving yields

$$
M^C_1(\theta_1,\theta_2)-M^N_1(\theta_1,\theta_2)-\left(c_1-\tilde{c}_1\right)\left(D^C_1(\theta_1,\theta_2)-\frac{2\left(2-b^2\right)}{\left(4-b^2\right)}D^N_1(\theta_1,\theta_2)\right)-\frac{\left(2-b^2\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}\left(c_1-\tilde{c}_1\right)^2\geq 0
$$

where the left hand side is a concave parabola in $c_1-\tilde{c}_1$. Let $t_1 := \frac{\left(2-b^2\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}$, $t_2 := D^C_1(\theta_1,\theta_2) - \frac{2\left(2-b^2\right)}{\left(4-b^2\right)}D^N_1(\theta_1,\theta_2)$ and $t_3 := M^C_1(\theta_1,\theta_2)-M^N_1(\theta_1,\theta_2)$. Note that $t_1>0$, and $t_3>0$ by $\delta^*(\theta_1,\theta_2)<1$ (which follows from Condition 4b). Using the quadratic formula, the above inequality holds if

$$
\frac{t_2-\sqrt{t_2^2+4t_1t_3}}{-2t_1}\geq c_1-\tilde{c}_1\geq\frac{t_2+\sqrt{t_2^2+4t_1t_3}}{-2t_1}
$$

$$
\iff \frac{t_2-\sqrt{t_2^2+4t_1t_3}}{2t_1}\leq \tilde{c}_1-c_1\leq\frac{t_2+\sqrt{t_2^2+4t_1t_3}}{2t_1}
$$

$$
\iff \frac{t_2-\sqrt{t_2^2+4t_1t_3}}{2t_1}\leq \tilde{c}_1-c_1\leq\frac{t_2+\sqrt{t_2^2+4t_1t_3}}{2t_1}
$$

$$\iff \frac{t_2 - \sqrt{t_2^2 + 4t_1t_3}}{2t_1} \le -\theta_1 c_1 \le \frac{t_2 + \sqrt{t_2^2 + 4t_1t_3}}{2t_1}$$

$$\iff \frac{1}{c_1}\frac{-t_2 + \sqrt{t_2^2 + 4t_1t_3}}{2t_1} \ge \theta_1 \ge \frac{1}{c_1}\frac{-t_2 - \sqrt{t_2^2 + 4t_1t_3}}{2t_1}.$$

Note that

$$\frac{1}{c_1}\frac{-t_2 - \sqrt{t_2^2 + 4t_1t_3}}{2t_1} \le 0$$

$$\iff -t_2 - \sqrt{t_2^2 + 4t_1t_3} \le 0$$

$$\iff -t_2 \le \sqrt{t_2^2 + 4t_1t_3}$$

which holds as $t_1 > 0$ and $t_3 > 0$. Thus, owner 1 does not wish to defect if

$$\theta_1 \le \frac{1}{c_1}\frac{-t_2 + \sqrt{t_2^2 + 4t_1t_3}}{2t_1}.$$

Routine derivations show that, when $\delta^*(\theta_1, \theta_2) < 1$ (which holds by Condition 4b), $\theta_1 \le \frac{1}{c_1}\frac{-t_2+\sqrt{t_2^2+4t_1t_3}}{2t_1}$ if and only if

$$\theta_1 \le \frac{b(1-\tilde{c}_2+a) + \sqrt{(2c_1 - b(1-\tilde{c}_2+a))^2 + 4K_1}}{2c_1}$$

where

$$K_1 = -\left(\frac{4\left(\left(2-b^2\right)(1-c_1) - b(1-\tilde{c}_2+a)\right)^2}{\left(4-b^2\right)^2} - (1-2c_1)(1-b(1-\tilde{c}_2+a))\right).$$

Owner 1 Case 2 and Case 3: When $(0,\theta_2) \in S_{2,Limit}$ or $(0,\theta_2) \in S_{2,Monop}$, the $(0,\theta_2)$ sub-game results in firm 1 not producing any output and, as a result, owner 1 earning 0 profit. Thus, it suffices to show

$$0 \le \pi_1^C(\theta_1,\theta_2) = \frac{1}{(1-b^2)}\left(1 - \frac{(1+\tilde{c}_1)}{2} - b\frac{(1-\tilde{c}_2+a)}{2}\right)\left(\frac{1+\tilde{c}_1}{2} - c_1\right).$$

The derivations in Case 1 for owner 1 show that $\delta^*(\theta_1,\theta_2) < 1$ and

$$\theta_1 \le \frac{b(1-\tilde{c}_2+a) + \sqrt{(2c_1 - b(1-\tilde{c}_2+a))^2 + 4K_1}}{2c_1}$$

imply that

$$\frac{\left(\left(2-b^2\right)(1-c_1) - b(1-\tilde{c}_2+a)\right)^2}{\left(4-b^2\right)^2(1-b^2)}$$

$$\le \pi_1^C(\theta_1,\theta_2) = \frac{1}{2(1-b^2)}(1-\tilde{c}_1 - b(1-\tilde{c}_2+a))\left(\frac{1+\tilde{c}_1}{2} - c_1\right)$$

which implies

$$\pi_1^C(\theta_1,\theta_2) \ge 0$$

as $\frac{\left(\left(2-b^2\right)(1-c_1)-b(1-\tilde{c}_2+a)\right)^2}{(4-b^2)^2(1-b^2)} \ge 0$. Thus, owner 1 has no incentive to defect.

Owner 2: Next, we show owner 2 has no incentive to defect in stage 0. Note that

$$\begin{aligned}\pi_2^C(\theta_1,\theta_2) &= D_2\left(p_1^C,p_2^C\right)\left(p_2^C-c_2\right)\\ &= \frac{1}{1-b^2}\left(1-b+a-\frac{1+\tilde{c}_2+a}{2}+b\left(\frac{1+\tilde{c}_1}{2}\right)\right)\left(\frac{1+\tilde{c}_2+a}{2}-c_2\right)\\ &= \frac{1}{2\left(1-b^2\right)}\left(2-2b+2a-\left(1+\tilde{c}_2+a\right)+b\left(1+\tilde{c}_1\right)\right)\left(\frac{1+\tilde{c}_2+a}{2}-c_2\right)\\ &= \frac{1}{2\left(1-b^2\right)}\left(1-\tilde{c}_2-b+a+b\tilde{c}_1\right)\left(\frac{1+\tilde{c}_2+a}{2}-c_2\right)\\ &= \frac{1}{2\left(1-b^2\right)}\left(1-\tilde{c}_2+a-b\left(1-\tilde{c}_1\right)\right)\left(\frac{1+\tilde{c}_2+a}{2}-c_2\right).\end{aligned}$$

As $\pi_2^N(\theta_1,\theta_2)$ is non-increasing in θ_2 (see Lemma 3 from the text appendix), it suffices to show

$$\pi_2^N(\theta_1,0)\leq\pi_2^C(\theta_1,\theta_2).$$

$\delta^*(\theta_1,\theta_2)<1$ implies that (by Lemma A.8)

$$(1-\tilde{c}_1)\left(1-\frac{\left(4-b^2\right)\sqrt{b^2+8}-b^3}{8}\right)<\tilde{c}_2-\tilde{c}_1-a<(1-\tilde{c}_1)\left(1-\frac{\left(4-b^2\right)\sqrt{b^2+8}+b^3}{16-6b^2}\right).$$

The above inequality implies that (by Lemma A.1)

$$(1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right)<\tilde{c}_2-\tilde{c}_1-a$$

which implies that

$$(1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right)<c_2-\tilde{c}_1-a.$$

Thus, there are three possible cases in the $(\theta_1,0)$-sub-game:

Case 1: $(\theta_1,0)\in S_{Both}$

Case 2: $(\theta_1,0)\in S_{1,Limit}$

Case 3: $(\theta_1,0)\in S_{1,Monop}$

where S_{Both}, $S_{1,Limit}$ and $S_{1,Monop}$ are defined in Subsection A.7.1.

Owner 2 Case 1: When $(\theta_1,0)\in S_{Both}$,

$$\pi_2^N\left(\theta_1,0\right)=\pi_{2,Both}^N\left(\theta_1,0\right)=\frac{\left(\left(2-b^2\right)\left(1-c_2+a\right)-b\left(1-\tilde{c}_1\right)\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}.$$

Owner 2 has no incentive to defect when

$$\pi_{2,Both}^{N}(\theta_1,0) \leq \pi_2^C(\theta_1,\theta_2)$$

$$\frac{\left(\left(2-b^2\right)(1-c_2+a)-b(1-\tilde{c}_1)\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)} \leq \frac{1}{2\left(1-b^2\right)}(1-\tilde{c}_2+a-b(1-\tilde{c}_1))\left(\frac{1+\tilde{c}_2+a}{2}-c_2\right)$$

$$\frac{\left(\left(2-b^2\right)(1-c_2+a)-b(1-\tilde{c}_1)\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)} \leq \frac{1}{2\left(1-b^2\right)}(1-\tilde{c}_2+a-b(1-\tilde{c}_1))\left(\frac{1+\tilde{c}_2+a}{2}-\tilde{c}_2\right) - (c_2-\tilde{c}_2)\frac{1}{2\left(1-b^2\right)}(1-\tilde{c}_2+a-b(1-\tilde{c}_1))$$

$$\frac{\left(\left(2-b^2\right)(1-\tilde{c}_2+a)+(\tilde{c}_2-c_2)\left(2-b^2\right)-b(1-\tilde{c}_1)\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)} \leq \frac{1}{2\left(1-b^2\right)}(1-\tilde{c}_2+a-b(1-\tilde{c}_1))\left(\frac{1+\tilde{c}_2+a}{2}-\tilde{c}_2\right) - (c_2-\tilde{c}_2)\frac{1}{2\left(1-b^2\right)}(1-\tilde{c}_2+a-b(1-\tilde{c}_1))$$

$$\frac{\left(\left(2-b^2\right)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1)\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)} \leq \frac{1}{2\left(1-b^2\right)}(1-\tilde{c}_2+a-b(1-\tilde{c}_1))\left(\frac{1+\tilde{c}_2+a}{2}-\tilde{c}_2\right)$$

$$-2(c_2-\tilde{c}_2)\left(2-b^2\right)\frac{\left(2-b^2\right)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1)}{\left(4-b^2\right)^2\left(1-b^2\right)} - (c_2-\tilde{c}_2)\frac{1}{2\left(1-b^2\right)}(1-\tilde{c}_2+a-b(1-\tilde{c}_1))$$

$$+\frac{\left(2-b^2\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}(c_2-\tilde{c}_2)^2$$

$$M_2^N(\theta_1,\theta_2)+\frac{\left(2-b^2\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}(c_2-\tilde{c}_2)^2 \leq M_2^C(\theta_1,\theta_2)$$

$$-2(c_2-\tilde{c}_2)\left(2-b^2\right)\frac{\left(2-b^2\right)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1)}{\left(4-b^2\right)^2\left(1-b^2\right)} - (c_2-\tilde{c}_2)\frac{1}{2\left(1-b^2\right)}(1-\tilde{c}_2+a-b(1-\tilde{c}_1))$$

$$-2(c_2-\tilde{c}_2)\left(2-b^2\right)\frac{\left(2-b^2\right)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1)}{\left(4-b^2\right)^2\left(1-b^2\right)} \leq M_2^C(\theta_1,\theta_2)-M_2^N(\theta_1,\theta_2)$$

$$(c_2-\tilde{c}_2)\frac{1}{2\left(1-b^2\right)}(1-\tilde{c}_2+a-b(1-\tilde{c}_1))$$

$$+\frac{\left(2-b^2\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}(c_2-\tilde{c}_2)^2$$

$$(c_2-\tilde{c}_2)\left(D_2^C(\theta_1,\theta_2)-\frac{2\left(2-b^2\right)}{\left(4-b^2\right)}D_2^N(\theta_1,\theta_2)\right) \leq M_2^C(\theta_1,\theta_2)-M_2^N(\theta_1,\theta_2).$$

$$+\frac{\left(2-b^2\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}(c_2-\tilde{c}_2)^2$$

Solving yields

$$M_2^C(\theta_1,\theta_2)-M_2^N(\theta_1,\theta_2)-(c_2-\tilde{c}_2)\left(D_1^C(\theta_1,\theta_2)-\frac{2\left(2-b^2\right)}{\left(4-b^2\right)}D_1^N(\theta_1,\theta_2)\right)-\frac{\left(2-b^2\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}(c_2-\tilde{c}_2)^2 \geq 0$$

where the left hand side is a concave parabola in $c_2-\tilde{c}_2$. Let $t_1 \equiv \frac{\left(2-b^2\right)^2}{\left(4-b^2\right)^2\left(1-b^2\right)}$, $t_2 \equiv D_2^C(\theta_1,\theta_2) - \frac{2\left(2-b^2\right)}{\left(4-b^2\right)}D_2^N(\theta_1,\theta_2)$ and $t_3 \equiv M_2^C(\theta_1,\theta_2)-M_2^N(\theta_1,\theta_2)$. Note that $t_1>0$, and $t_3>0$ by $\delta^*(\theta_1,\theta_2)<1$. Using the quadratic formula, the above inequality holds if

$$\frac{t_2-\sqrt{t_2^2+4t_1t_3}}{-2t_1} \geq c_2-\tilde{c}_2 \geq \frac{t_2+\sqrt{t_2^2+4t_1t_3}}{-2t_1}$$

$$\iff \frac{t_2-\sqrt{t_2^2+4t_1t_3}}{2t_1} \leq \tilde{c}_2-c_2 \leq \frac{t_2+\sqrt{t_2^2+4t_1t_3}}{2t_1}$$

$$\iff \frac{t_2-\sqrt{t_2^2+4t_1t_3}}{2t_1} \leq \tilde{c}_2-c_2 \leq \frac{t_2+\sqrt{t_2^2+4t_1t_3}}{2t_1}$$

$$\iff \frac{t_2 - \sqrt{t_2^2 + 4t_1 t_3}}{2t_1} \leq -\theta_2 c_2 \leq \frac{t_2 + \sqrt{t_2^2 + 4t_1 t_3}}{2t_1}$$

$$\iff \frac{1}{c_2} \frac{-t_2 + \sqrt{t_2^2 + 4t_1 t_3}}{2t_1} \geq \theta_2 \geq \frac{1}{c_2} \frac{-t_2 - \sqrt{t_2^2 + 4t_1 t_3}}{2t_1}.$$

Note that

$$\frac{1}{c_2} \frac{-t_2 - \sqrt{t_2^2 + 4t_1 t_3}}{2t_1} \leq 0$$

$$\iff -t_2 - \sqrt{t_2^2 + 4t_1 t_3} \leq 0$$

$$\iff -t_2 \leq \sqrt{t_2^2 + 4t_1 t_3}$$

which holds as $t_1 > 0$ and $t_3 > 0$. Thus, owner 2 does not wish to defect if

$$\theta_2 \leq \frac{1}{c_2} \frac{-t_2 + \sqrt{t_2^2 + 4t_1 t_3}}{2t_1}.$$

Routine derivations show that, when $\delta^*(\theta_1, \theta_2) < 1$, $\theta_2 \leq \frac{1}{c_2} \frac{-t_2 + \sqrt{t_2^2 + 4t_1 t_3}}{2t_1}$ if and only if

$$\theta_2 \leq \frac{b(1 - \tilde{c}_1) + \sqrt{(2c_2 - b(1 - \tilde{c}_1))^2 + 4K_2}}{2c_2}$$

where

$$K_2 = -\left(\frac{4\left(\left(2 - b^2\right)(1 - c_2 + a) - b(1 - \tilde{c}_1)\right)^2}{\left(4 - b^2\right)^2} - (1 - 2c_2 + a)(1 + a - b(1 - \tilde{c}_1)) \right).$$

Owner 2 Case 2 and Case 3: When $(\theta_1, 0) \in S_{1,Limit}$ or $(\theta_1, 0) \in S_{1,Monop}$, the $(\theta_1, 0)$ sub-game results in firm 2 not producing any output and, as a result, owner 2 earning 0 profit. Thus, it suffices to show

$$0 \leq \pi_2^C(\theta_1, \theta_2) = \frac{1}{2(1 - b^2)} (1 - \tilde{c}_2 + a - b(1 - \tilde{c}_1)) \left(\frac{1 + \tilde{c}_2 + a}{2} - c_2 \right).$$

The derivations in Case 1 for owner 2 show that $\delta^*(\theta_1, \theta_2) < 1$ and

$$\theta_2 \leq \frac{b(1 - \tilde{c}_1) + \sqrt{(2c_2 - b(1 - \tilde{c}_1))^2 + 4K_2}}{2c_2}$$

imply that

$$\frac{\left(\left(2 - b^2\right)(1 - c_2 + a) - b(1 - \tilde{c}_1)\right)^2}{\left(4 - b^2\right)^2 (1 - b^2)}$$

$$\leq \pi_2^C(\theta_1, \theta_2) = \frac{1}{2(1 - b^2)} (1 - \tilde{c}_2 + a - b(1 - \tilde{c}_1)) \left(\frac{1 + \tilde{c}_2 + a}{2} - c_2 \right)$$

which implies

$$\pi_2^C(\theta_1, \theta_2) \geq 0$$

as $\frac{\left(\left(2-b^2\right)(1-c_2+a)-b(1-\tilde{c}_1)\right)^2}{(4-b^2)^2(1-b^2)} \geq 0$. Thus, owner 2 has no incentive to defect. □

A.6 Sales Weights Facilitating Collusion, Enhancing Firm Profit and Reducing Consumer Surplus

In this subsection, we provide a sufficient condition which ensures that there exists sales weights θ_1 and θ_2 which facilitate collusion, enhance firm profit and reduce consumer surplus. Note that, by Lemma A.8, $\delta^*(0,0) < 1$ if and only if

$$(1-c_1)\left(1-\frac{\left(4-b^2\right)\sqrt{b^2+8}-b^3}{8}\right) < c_2 - c_1 - a < (1-c_1)\left(1-\frac{\left(4-b^2\right)\sqrt{b^2+8}+b^3}{16-6b^2}\right).$$

Thus, $\delta^*(0,0) < 1$ if and only if the asymmetry between firms is sufficiently moderate.

Proposition A.1. *Suppose $c_2 - a \neq c_1$ and $\delta^*(0,0) < 1$. Then there exist $\theta_1, \theta_2 \in [0,1]$ such that*
i) $\delta^*(\theta_1,\theta_2) < \delta^*(0,0)$,
ii) $\pi_1^C(\theta_1,\theta_2) > \pi_1^N(0,0)$ and $\pi_2^C(\theta_1,\theta_2) > \pi_2^N(0,0)$, and
iii) $CS^C(\theta_1,\theta_2) < CS^N(0,0)$.

Proof. The proof proceeds by constructing a (θ_1,θ_2) such that part (i), (ii) and (iii) hold.[9] Without loss of generality, assume $c_2 - a > c_1$. Let $\theta_1 = 0$ and let $\theta_2 = \epsilon$ where $\epsilon > 0$ is small. Note that Condition 1 holds by

$$|c_2(1-\theta_2) - a - c_1(1-\theta_1)| = |c_2(1-\epsilon) - a - c_1| < c_2 - a - c_1$$

for sufficiently small ϵ. Thus, part (i) holds.

Next, consider part (ii). First, note that $\delta^*(0,0) < 1$ implies both firms are active in the Nash equilibrium when $\theta_1 = \theta_2 = 0$. Additionally, $\delta^*(0,0) < 1$ implies $M_1^N(0,0) < M_1^C(0,0)$ and $M_2^N(0,0) < M_2^C(0,0)$ (see the proof of Lemma A.8). This implies that $M_1^N(0,0) = \pi_1^N(0,0) < \pi_1^C(0,0) = M_1^C(0,0)$ and $M_2^N(0,0) = \pi_2^N(0,0) < \pi_2^C(0,0) = M_2^C(0,0)$. By continuity of $\pi_2^C(\theta_1,\theta_2)$, $\pi_2^C(0,0) > \pi_2^N(0,0)$ implies $\pi_2^C(\theta_1,\theta_2) = \pi_2^C(0,\epsilon) > \pi_2^N(0,0)$ for sufficiently small ϵ. Thus, $\pi_1^C(\theta_1,\theta_2) = \pi_1^C(0,0) > \pi_1^N(0,0)$ and $\pi_2^C(\theta_1,\theta_2) = \pi_2^C(0,\epsilon) > \pi_2^N(0,0)$ and part (ii) holds for both firms.

Part (iii) holds, by the proof of Proposition 5, as

$$\theta_1 = 0 < \frac{1}{c_1}\left(\frac{2b(1-c_2) + 2ab + b^2(1-c_1)}{4-b^2}\right)$$

and

$$\theta_2 = \epsilon < \frac{1}{c_2}\left(\frac{2b(1-c_1) + b^2(1-c_2+a)}{4-b^2}\right)$$

for sufficiently small ϵ. □

A.7 Additional Strategic Delegation Results

In this section, we present additional details and results related to strategic delegation or owners' endogenous choice of θ.

A.7.1 Preliminary Results

In this subsection, we explore owner profits in the Nash equilibrium (i.e., when managers compete when setting prices) and provide a number of intermediary results employed in other sections. Following Zanchettin (2006), note that, in the Nash equilibrium, there are five cases to consider:

Case 1: Both firms active in the Nash equilibrium. This case occurs if $(\theta_1,\theta_2) \in S_{Both}$ where

$$S_{Both} = \left\{(\theta_1,\theta_2) \in [0,1]^2 : (1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right) > \tilde{c}_2 - \tilde{c}_1 - a > (1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right)\right\}.$$

[9] The set of (θ_1,θ_2) that satisfy conditions i, ii and iii is not, in general, a singleton and can include a wide range of (θ_1,θ_2) values. See the figures in the main text.

Case 2: Firm 1 Inactive with Firm 2 Limit Pricing. This case occurs if $(\theta_1, \theta_2) \in S_{2,Limit}$ where

$$S_{2,Limit} = \left\{ (\theta_1, \theta_2) \in [0,1]^2 : \left(1 - \frac{2}{b}\right)(1 - \tilde{c}_1) < \tilde{c}_2 - \tilde{c}_1 - a < (1 - \tilde{c}_1)\left(1 - \frac{2 - b^2}{b}\right) \right\}.$$

Case 3: Firm 2 Inactive with Firm 1 Limit Pricing. This case occurs if $(\theta_1, \theta_2) \in S_{1,Limit}$ where

$$S_{1,Limit} = \left\{ (\theta_1, \theta_2) \in [0,1]^2 : (1 - \tilde{c}_1)\left(1 - \frac{b}{2 - b^2}\right) < \tilde{c}_2 - \tilde{c}_1 - a < \left(1 - \frac{b}{2}\right)(1 - \tilde{c}_1) \right\}.$$

Case 4: Firm 2 Inactive with Firm 1 Monopoly Pricing. This case occurs if $(\theta_1, \theta_2) \in S_{1,Monop}$ where

$$S_{1,Monop} = \left\{ (\theta_1, \theta_2) \in [0,1]^2 : \tilde{c}_2 - \tilde{c}_1 - a > \left(1 - \frac{b}{2}\right)(1 - \tilde{c}_1) \right\}.$$

Case 5: Firm 1 Inactive with Firm 2 Monopoly Pricing. This case occurs if $(\theta_1, \theta_2) \in S_{2,Monop}$ where

$$S_{2,Monop} = \left\{ (\theta_1, \theta_2) \in [0,1]^2 : \left(1 - \frac{2}{b}\right)(1 - \tilde{c}_1) > \tilde{c}_2 - \tilde{c}_1 - a \right\}.$$

Firm 1: Firm 1's profit in the Nash equilibrium of the delegation game (as a function of θ_1 and θ_2) is

$$\pi_1^N(\theta_1, \theta_2) = \begin{cases} 0 & \text{if } (1 - \tilde{c}_1)\left(1 - \frac{2-b^2}{b}\right) \geq x - a \\ \pi_{1,Both}^N(\theta_1, \theta_2) & \text{if } (1 - \tilde{c}_1)\left(1 - \frac{b}{2-b^2}\right) \geq x - a > (1 - \tilde{c}_1)\left(1 - \frac{2-b^2}{b}\right) \\ \pi_{1,Limit}^N(\theta_1, \theta_2) & \text{if } \left(1 - \frac{b}{2}\right)(1 - \tilde{c}_1) \geq x - a > (1 - \tilde{c}_1)\left(1 - \frac{b}{2-b^2}\right) \\ \pi_{1,Monop}^N(\theta_1, \theta_2) & \text{if } x - a > \left(1 - \frac{b}{2}\right)(1 - \tilde{c}_1) \end{cases}$$

where $\pi_{1,Both}^N(\theta_1, \theta_2)$ denotes profit if both firms are active in the Nash equilibrium, $\pi_{1,Limit}^N(\theta_1, \theta_2)$ denotes profit if only Firm 1 is active and the limit pricing equilibrium occurs Zanchettin (2006), and $\pi_{1,Monop}^N(\theta_1, \theta_2)$ denotes profit if only Firm 1 is active and sets the monopoly price Zanchettin (2006).

Derivation of $\pi_{1,Both}^N(\theta_1, \theta_2)$: Firm 1's profit when both firms are active in the Nash equilibrium is

$$\pi_{1,Both}^N(\theta_1, \theta_2) = D_1\left(p_1^N(\theta_1, \theta_2), p_2^N(\theta_1, \theta_2)\right)\left(p_1^N(\theta_1, \theta_2) - c_1\right)$$

where

$$p_1^N(\theta_1, \theta_2) = \frac{2 - b + 2\tilde{c}_1 - ab + b\tilde{c}_2 - b^2}{4 - b^2}$$

and

$$p_2^N(\theta_1, \theta_2) = \frac{2 - b + 2\tilde{c}_2 + 2a + b\tilde{c}_1 - ab^2 - b^2}{4 - b^2}.$$

Simplifying yields

$$\begin{aligned}
\pi_{1,Both}^N(\theta_1,\theta_2) &= D_1\left(p_1^N(\theta_1,\theta_2), p_2^N(\theta_1,\theta_2)\right)\left(p_1^N(\theta_1,\theta_2)-c_1\right) \\
&= \frac{1}{1-b^2}\left[1-b-ba-p_1^N(\theta_1,\theta_2)+bp_2^N(\theta_1,\theta_2)\right]\left(p_1^N(\theta_1,\theta_2)-c_1\right) \\
&= \frac{1}{1-b^2}\left[1-b-ba-\frac{2-b+2\tilde{c}_1-ab+b\tilde{c}_2-b^2}{4-b^2}+b\frac{2-b+2\tilde{c}_2+2a+b\tilde{c}_1-ab^2-b^2}{4-b^2}\right] \\
&\times\left(p_1^N(\theta_1,\theta_2)-c_1\right) \\
&= \frac{1}{1-b^2}\left[1-b-ba-\frac{2-b+2\tilde{c}_1-ab+b\tilde{c}_2-b^2-\left(2b-b^2+2b\tilde{c}_2+2ab+b^2\tilde{c}_1-ab^3-b^3\right)}{4-b^2}\right] \\
&\times\left(p_1^N(\theta_1,\theta_2)-c_1\right) \\
&= \frac{1}{1-b^2}\left[1-b-ba-\frac{2-3b+\left(2-b^2\right)\tilde{c}_1-3ab-b\tilde{c}_2+ab^3+b^3}{4-b^2}\right]\left(p_1^N(\theta_1,\theta_2)-c_1\right) \\
&= \frac{1}{1-b^2}\left[\frac{4-4b-4ba-b^2+b^3+b^3a-\left(2-3b+\left(2-b^2\right)\tilde{c}_1-3ab-b\tilde{c}_2+ab^3+b^3\right)}{4-b^2}\right]\left(p_1^N(\theta_1,\theta_2)-c_1\right) \\
&= \frac{1}{1-b^2}\left[\frac{2-b-ba-b^2-\left(\left(2-b^2\right)\tilde{c}_1-b\tilde{c}_2\right)}{4-b^2}\right]\left(p_1^N(\theta_1,\theta_2)-c_1\right) \\
&= \frac{1}{1-b^2}\left[\frac{\left(\left(2-b^2\right)\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)\right)}{4-b^2}\right]\left(p_1^N(\theta_1,\theta_2)-c_1\right) \\
&= \frac{1}{1-b^2}\left[\frac{\left(\left(2-b^2\right)\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)\right)}{4-b^2}\right]\left(\frac{2-b+2\tilde{c}_1-ab+b\tilde{c}_2-b^2}{4-b^2}-c_1\right) \\
&= \frac{1}{1-b^2}\left[\frac{\left(\left(2-b^2\right)\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)\right)}{4-b^2}\right]\left(\frac{2-b+2\tilde{c}_1-ab+b\tilde{c}_2-b^2-4c_1+b^2c_1}{4-b^2}\right) \\
&= \frac{1}{1-b^2}\left[\frac{\left(\left(2-b^2\right)\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)\right)}{4-b^2}\right]\left(\frac{\left(2-b^2\right)\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)+\left(4-b^2\right)\left(\tilde{c}_1-c_1\right)}{4-b^2}\right) \\
&= \frac{1}{1-b^2}\left[\frac{\left(\left(2-b^2\right)\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)\right)}{4-b^2}\right]\left(\frac{\left(2-b^2\right)\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)}{4-b^2}+\left(\tilde{c}_1-c_1\right)\right)
\end{aligned}$$

Derivation of $\pi_{1,Limit}^N(\theta_1,\theta_2)$: The limit pricing equilibrium refers to a case where the perceived asymmetry between the firms is sufficiently large that firm 2 does not produce. However, the perceived asymmetry is sufficiently small that firm 1's price is constrained by firm 2 Zanchettin (2006). This case occurs when

$$\left(1-\tilde{c}_1\right)\left(1-\frac{b}{2-b^2}\right) < \tilde{c}_2-\tilde{c}_1-a < \left(1-\frac{b}{2}\right)\left(1-\tilde{c}_1\right) \tag{14}$$

In this case, firm 1 sets a price of $p_1^I(\theta_1,\theta_2)$ where

$$D_2(p_1^I,\tilde{c}_2) = 1-b+a-\tilde{c}_2+bp_1^I(\theta_1,\theta_2) = 0$$

or

$$p_1^I(\theta_1,\theta_2) = \frac{b+\tilde{c}_2-a-1}{b}.$$

Note that $p_1^I(\theta_1,\theta_2) < p_1^M(\theta_1,\theta_2) = \frac{1+\tilde{c}_1}{2}$ as

$$\begin{aligned}
p_{I,1}(\theta_1,\theta_2) = \frac{b+\tilde{c}_2-a-1}{b} &< \frac{1+\tilde{c}_1}{2} = p_1^M(\theta_1,\theta_2) \\
2b+2\tilde{c}_2-2a-2 &< b+b\tilde{c}_1 \\
b+2\tilde{c}_2-2a-2 &< b\tilde{c}_1 \\
b+2\tilde{c}_2-2a-2-2\tilde{c}_1 &< b\tilde{c}_1-2\tilde{c}_1 \\
2\tilde{c}_2-2a-2\tilde{c}_1 &< b\tilde{c}_1-2\tilde{c}_1+2-b \\
2\tilde{c}_2-2a-2\tilde{c}_1 &< (2-b)(1-\tilde{c}_1) \\
\tilde{c}_2-\tilde{c}_1-a &< \left(1-\frac{b}{2}\right)(1-\tilde{c}_1)
\end{aligned}$$

where the last inequality holds by inequality (14). Profits in this case are

$$\begin{aligned}
\pi_{1,Limit}^N(\theta_1,\theta_2) &= (1-p_1^I)(p_1^I-c_1) \\
&= \left(1-\frac{b+\tilde{c}_2-a-1}{b}\right)\left(\frac{b+\tilde{c}_2-a-1}{b}-c_1\right) \\
&= \left(\frac{b-(b+\tilde{c}_2-a-1)}{b}\right)\left(\frac{b+\tilde{c}_2-a-1}{b}-c_1\right) \\
&= \left(\frac{1-\tilde{c}_2+a}{b}\right)\left(\frac{b+\tilde{c}_2-a-1}{b}-c_1\right)
\end{aligned}$$

Derivation of $\pi_{1,Monop}^N(\theta_1,\theta_2)$: When the perceived asymmetry is particularly large, firm 2 is inactive and firm 1 sets the monopoly price. Thus, firm 1's pricing is unconstrained by firm 2. Following Zanchettin (2006), this case occurs when

$$\tilde{c}_2-\tilde{c}_1-a > \left(1-\frac{b}{2}\right)(1-\tilde{c}_1). \tag{15}$$

Note that

$$\begin{aligned}
p_{I,1}(\theta_1,\theta_2) = \frac{b+\tilde{c}_2-a-1}{b} &> \frac{1+\tilde{c}_1}{2} = p_1^M(\theta_1,\theta_2) \\
2b+2\tilde{c}_2-2a-2 &> b+b\tilde{c}_1 \\
b+2\tilde{c}_2-2a-2 &> b\tilde{c}_1 \\
b+2\tilde{c}_2-2a-2-2\tilde{c}_1 &> b\tilde{c}_1-2\tilde{c}_1 \\
2\tilde{c}_2-2a-2\tilde{c}_1 &> b\tilde{c}_1-2\tilde{c}_1+2-b \\
2\tilde{c}_2-2a-2\tilde{c}_1 &> (2-b)(1-\tilde{c}_1) \\
\tilde{c}_2-\tilde{c}_1-a &> \left(1-\frac{b}{2}\right)(1-\tilde{c}_1).
\end{aligned}$$

where the last inequality holds by inequality (15).

Firm 1's profit in this case is

$$\pi_{1,Monop}^N(\theta_1,\theta_2) = \left(1-p_1^M(\theta_1,\theta_2)\right)\left(p_1^M(\theta_1,\theta_2)-c_1\right)$$

where

$$p_1^M(\theta_1,\theta_2) = \frac{1+\tilde{c}_1}{2}.$$

Simplifying yields

$$\begin{aligned}
\pi^N_{1,Monop}(\theta_1,\theta_2) &= \left(1-\frac{1+\tilde{c}_1}{2}\right)\left(\frac{1+\tilde{c}_1}{2}-c_1\right)\\
&= \left(\frac{1-\tilde{c}_1}{2}\right)\left(\frac{1+\tilde{c}_1-2c_1}{2}\right)\\
&= \left(\frac{1-\tilde{c}_1}{2}\right)\left(\frac{1+c_1(1-\theta_1)-2c_1}{2}\right)\\
&= \left(\frac{1-c_1+c_1\theta_1}{2}\right)\left(\frac{1+c_1-\theta_1c_1-2c_1}{2}\right)\\
&= \left(\frac{1-c_1+c_1\theta_1}{2}\right)\left(\frac{1-c_1-\theta_1c_1}{2}\right)\\
&= \frac{(1-c_1)^2-(c_1\theta_1)^2}{4}
\end{aligned}$$

Continuity of $\pi^N_1(\theta_1,\theta_2)$: The following lemma establishes the continuity of $\pi^N_1(\theta_1,\theta_2)$ in θ_1.

Lemma A.9. *$\pi^N_1(\theta_1,\theta_2)$ is continuous in θ_1.*

Proof. It suffices to show that

Part 1: $\pi^N_{1,Both}(\theta_1,\theta_2)=0$ when $(1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right)=x-a$

Part 2: $\pi^N_{1,Both}(\theta_1,\theta_2)=\pi^N_{1,Limit}(\theta_1,\theta_2)$ when $(1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right)=x-a$

Part 3: $\pi^N_{1,Limit}(\theta_1,\theta_2)=\pi^N_{1,Monop}(\theta_1,\theta_2)$ when $\left(1-\frac{b}{2}\right)(1-\tilde{c}_1)=x-a$

Part 1: Note that

$$\begin{aligned}
\tilde{c}_2-\tilde{c}_1-a &= (1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right)\\
\tilde{c}_2-\tilde{c}_1-a-(1-\tilde{c}_1) &= \left(-\frac{2-b^2}{b}\right)(1-\tilde{c}_1)\\
\tilde{c}_2-a-1 &= \left(-\frac{2-b^2}{b}\right)(1-\tilde{c}_1)\\
0 &= \left(-\left(2-b^2\right)\right)(1-\tilde{c}_1)+(1-\tilde{c}_2+a)\,b\\
0 &= \left(2-b^2\right)(1-\tilde{c}_1)-(1-\tilde{c}_2+a)\,b\\
\implies 0 &= D_1(p^N_1(\theta_1,\theta_2),p^N_2(\theta_1,\theta_2))
\end{aligned}$$

which implies

$$\pi^N_{1,Both}(\theta_1,\theta_2)=0$$

when $(1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right)=x-a$.

Part 2: Note that

$$\begin{aligned}
(1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right) &= \tilde{c}_2-\tilde{c}_1-a\\
(1-\tilde{c}_1)\left(2-b^2-b\right) &= \left(2-b^2\right)(\tilde{c}_2-\tilde{c}_1-a)\\
(1-\tilde{c}_1)(-b) &= \left(2-b^2\right)(\tilde{c}_2-\tilde{c}_1-a-1+\tilde{c}_1)\\
(1-\tilde{c}_1)(-b) &= \left(2-b^2\right)(\tilde{c}_2-a-1)\\
0 &= b\,(1-\tilde{c}_1)-\left(2-b^2\right)(1-\tilde{c}_2+a)\,.\\
1-\tilde{c}_2+a &= \frac{b}{2-b^2}(1-\tilde{c}_1)
\end{aligned}$$

Thus, $1-\tilde{c}_2+a=\frac{b}{2-b^2}\left(1-\tilde{c}_1\right)$ and $\frac{(1-\tilde{c}_2+a)}{b}=\frac{1}{2-b^2}\left(1-\tilde{c}_1\right)$ hold when $\left(1-\tilde{c}_1\right)\left(1-\frac{b}{2-b^2}\right)=\tilde{c}_2-\tilde{c}_1-a$. Substituting $\frac{(1-\tilde{c}_2+a)}{b}=\frac{1}{2-b^2}\left(1-\tilde{c}_1\right)$ into $\pi^N_{1,Limit}(\theta_1,\theta_2)$ yields

$$\begin{aligned}
\pi^N_{1,Limit}(\theta_1,\theta_2) &= \left(\frac{1-\tilde{c}_2+a}{b}\right)\left(\frac{b+\tilde{c}_2-a-1}{b}-c_1\right) \\
&= \frac{1}{2-b^2}\left(1-\tilde{c}_1\right)\left(1-\frac{1}{2-b^2}\left(1-\tilde{c}_1\right)-c_1\right) \\
&= \frac{1}{2-b^2}\left(1-\tilde{c}_1\right)\left(1-\tilde{c}_1-\frac{1}{2-b^2}\left(1-\tilde{c}_1\right)+\tilde{c}_1-c_1\right) \\
&= \frac{1}{2-b^2}\left(1-\tilde{c}_1\right)\left(\left(1-\frac{1}{2-b^2}\right)\left(1-\tilde{c}_1\right)+\tilde{c}_1-c_1\right) \\
&= \frac{1}{2-b^2}\left(1-\tilde{c}_1\right)\left(\left(\frac{1-b^2}{2-b^2}\right)\left(1-\tilde{c}_1\right)+\tilde{c}_1-c_1\right)
\end{aligned}$$

Substituting $\left(1-\tilde{c}_2+a\right)=\frac{b}{2-b^2}\left(1-\tilde{c}_1\right)$ into $\pi^N_{1,Both}(\theta_1,\theta_2)$ yields

$$\begin{aligned}
\pi^N_{1,Both}(\theta_1,\theta_2) &= D_1\left(p^N_1(\theta_1,\theta_2),p^N_2(\theta_1,\theta_2)\right)\left(p^N_1(\theta_1,\theta_2)-c_1\right) \\
&= \frac{1}{1-b^2}\left[\frac{\left(\left(2-b^2\right)\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)\right)}{4-b^2}\right]\left(\frac{\left(2-b^2\right)\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)}{4-b^2}+\left(\tilde{c}_1-c_1\right)\right) \\
&= \frac{1}{1-b^2}\left[\frac{\left(\left(2-b^2\right)\left(1-\tilde{c}_1\right)-b\left(\frac{b}{2-b^2}\left(1-\tilde{c}_1\right)\right)\right)}{4-b^2}\right] \\
&\times\left(\frac{\left(2-b^2\right)\left(1-\tilde{c}_1\right)-b\left(\frac{b}{2-b^2}\left(1-\tilde{c}_1\right)\right)}{4-b^2}+\left(\tilde{c}_1-c_1\right)\right) \\
&= \frac{1}{\left(1-b^2\right)\left(2-b^2\right)}\left[\frac{\left(\left(2-b^2\right)^2\left(1-\tilde{c}_1\right)-b^2\left(1-\tilde{c}_1\right)\right)}{4-b^2}\right] \\
&\times\left(\frac{1}{2-b^2}\frac{\left(2-b^2\right)^2\left(1-\tilde{c}_1\right)-b^2\left(1-\tilde{c}_1\right)}{4-b^2}+\left(\tilde{c}_1-c_1\right)\right) \\
&= \frac{1}{\left(1-b^2\right)\left(2-b^2\right)}\left[\frac{\left(4-5b^2+b^4\right)\left(1-\tilde{c}_1\right)}{4-b^2}\right]\left(\frac{1}{2-b^2}\frac{\left(4-5b^2+b^4\right)\left(1-\tilde{c}_1\right)}{4-b^2}+\left(\tilde{c}_1-c_1\right)\right) \\
&= \frac{1}{\left(1-b^2\right)\left(2-b^2\right)}\left[\frac{\left(4-b^2\right)\left(1-b^2\right)\left(1-\tilde{c}_1\right)}{4-b^2}\right]\left(\frac{1}{2-b^2}\frac{\left(4-b^2\right)\left(1-b^2\right)\left(1-\tilde{c}_1\right)}{4-b^2}+\left(\tilde{c}_1-c_1\right)\right) \\
&= \frac{1}{\left(2-b^2\right)}\left(1-\tilde{c}_1\right)\left(\frac{1}{2-b^2}\left(1-b^2\right)\left(1-\tilde{c}_1\right)+\tilde{c}_1-c_1\right)
\end{aligned}$$

which implies $\pi^N_{1,Both}(\theta_1,\theta_2)=\pi^N_{1,Limit}(\theta_1,\theta_2)$ when $\left(1-\tilde{c}_1\right)\left(1-\frac{b}{2-b^2}\right)=\tilde{c}_2-\tilde{c}_1-a$.

Part 3: It suffices to show that $p_1^I(\theta_1,\theta_2) = p_1^M(\theta_1,\theta_2)$ when $\left(1-\frac{b}{2}\right)(1-\tilde{c}_1) = \tilde{c}_2 - \tilde{c}_1 - a$.

$$\begin{aligned}
p_{I,1}(\theta_1,\theta_2) = \frac{b+\tilde{c}_2-a-1}{b} &= \frac{1+\tilde{c}_1}{2} = p_1^M(\theta_1,\theta_2) \\
\iff 2b+2\tilde{c}_2-2a-2 &= b+b\tilde{c}_1 \\
\iff b+2\tilde{c}_2-2a-2 &= b\tilde{c}_1 \\
\iff b+2\tilde{c}_2-2a-2-2\tilde{c}_1 &= b\tilde{c}_1-2\tilde{c}_1 \\
\iff 2\tilde{c}_2-2a-2\tilde{c}_1 &= b\tilde{c}_1-2\tilde{c}_1+2-b \\
\iff 2\tilde{c}_2-2a-2\tilde{c}_1 &= (2-b)(1-\tilde{c}_1) \\
\iff \tilde{c}_2-\tilde{c}_1-a &= \left(1-\frac{b}{2}\right)(1-\tilde{c}_1).
\end{aligned}$$

Thus, $\pi_{1,Limit}^N(\theta_1,\theta_2) = \pi_{1,Monop}^N(\theta_1,\theta_2)$ when $\left(1-\frac{b}{2}\right)(1-\tilde{c}_1) = \tilde{c}_2-\tilde{c}_1-a$. □

Firm 2: Firm 2's profit in the Nash equilibrium of the delegation game (as a function of θ_1 and θ_2) is

$$\pi_2^N(\theta_1,\theta_2) = \begin{cases} 0 & \text{if } (1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right) \geq x-a \\ \pi_{2,Both}^N(\theta_1,\theta_2) & \text{if } (1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right) > x-a \geq (1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right) \\ \pi_{2,Limit}^N(\theta_1,\theta_2) & \text{if } (1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right) > x-a \geq (1-\tilde{c}_1)\left(1-\frac{2}{b}\right) \\ \pi_{2,Monop}^N(\theta_1,\theta_2) & \text{if } x-a < \left(1-\frac{2}{b}\right)(1-\tilde{c}_1) \end{cases}$$

where $\pi_{2,Both}^N(\theta_1,\theta_2)$ denotes profit if both firms are active in the Nash equilibrium, $\pi_{2,Limit}^N(\theta_1,\theta_2)$ denotes profit if only firm 2 is active and the limit pricing equilibrium occurs Zanchettin (2006), and $\pi_{2,Monop}^N(\theta_1,\theta_2)$ denotes profit if only Firm 2 is active and sets the monopoly price Zanchettin (2006).

Derivation of $\pi_{2,Both}^N(\theta_1,\theta_2)$: Firm 2's profit when both firms are active in the Nash equilibrium is

$$\pi_{2,Both}^N(\theta_1,\theta_2) = D_2\left(p_1^N(\theta_1,\theta_2), p_2^N(\theta_1,\theta_2)\right)\left(p_2^N(\theta_1,\theta_2) - c_2\right)$$

where

$$p_1^N(\theta_1,\theta_2) = \frac{2-b+2\tilde{c}_1-ab+b\tilde{c}_2-b^2}{4-b^2}$$

and

$$p_2^N(\theta_1,\theta_2) = \frac{2-b+2\tilde{c}_2+2a+b\tilde{c}_1-ab^2-b^2}{4-b^2}.$$

Simplifying yields

$$\begin{aligned}
\pi^N_{2,Both}(\theta_1,\theta_2) &= D_2\left(p^N_1(\theta_1,\theta_2), p^N_2(\theta_1,\theta_2)\right)\left(p^N_2(\theta_1,\theta_2)-c_2\right) \\
&= \frac{1}{1-b^2}\left[1-b+a-p^N_2(\theta_1,\theta_2)+bp^N_1(\theta_1,\theta_2)\right]\left(p^N_2(\theta_1,\theta_2)-c_2\right) \\
&= \frac{1}{1-b^2}\left[1-b+a-\frac{2-b+2\tilde{c}_2+2a+b\tilde{c}_1-ab^2-b^2}{4-b^2}+b\frac{2-b+2\tilde{c}_1-ab+b\tilde{c}_2-b^2}{4-b^2}\right] \\
&\times\left(p^N_2(\theta_1,\theta_2)-c_2\right) \\
&= \frac{1}{1-b^2}\left[1-b+a-\frac{2-b+2\tilde{c}_2+2a+b\tilde{c}_1-ab^2-b^2-\left(2b-b^2+2b\tilde{c}_1-ab^2+b^2\tilde{c}_2-b^3\right)}{4-b^2}\right] \\
&\times\left(p^N_2(\theta_1,\theta_2)-c_2\right) \\
&= \frac{1}{1-b^2}\left[1-b+a-\frac{2-3b+\left(2-b^2\right)\tilde{c}_2+2a-b\tilde{c}_1+b^3}{4-b^2}\right]\left(p^N_2(\theta_1,\theta_2)-c_2\right) \\
&= \frac{1}{1-b^2}\left[\frac{4-4b+4a-b^2+b^3-b^2a-\left(2-3b+\left(2-b^2\right)\tilde{c}_2+2a-b\tilde{c}_1+b^3\right)}{4-b^2}\right]\left(p^N_2(\theta_1,\theta_2)-c_2\right) \\
&= \frac{1}{1-b^2}\left[\frac{2-b+2a-b^2-b^2a-\left(\left(2-b^2\right)\tilde{c}_2-b\tilde{c}_1\right)}{4-b^2}\right]\left(p^N_2(\theta_1,\theta_2)-c_2\right) \\
&= \frac{1}{1-b^2}\left[\frac{\left(\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)\right)}{4-b^2}\right]\left(p^N_2(\theta_1,\theta_2)-c_2\right) \\
&= \frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)}{4-b^2}\right]\left(\frac{2-b+2\tilde{c}_2+2a+b\tilde{c}_1-ab^2-b^2}{4-b^2}-c_2\right) \\
&= \frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)}{4-b^2}\right]\left(\frac{2-b+2\tilde{c}_2+2a+b\tilde{c}_1-ab^2-b^2-\left(4-b^2\right)c_2}{4-b^2}\right) \\
&= \frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)}{4-b^2}\right]\left(\frac{\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)+\left(4-b^2\right)\left(\tilde{c}_2-c_2\right)}{4-b^2}\right) \\
&= \frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)}{4-b^2}\right]\left(\frac{\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)}{4-b^2}+\left(\tilde{c}_2-c_2\right)\right)
\end{aligned}$$

Derivation of $\pi^N_{2,Limit}(\theta_1,\theta_2)$: The limit pricing equilibrium refers to a case where the perceived asymmetry between the firms is sufficiently large that Firm 1 does not produce. However, the perceived asymmetry is sufficiently small that firm 2's price is constrained by firm 1 Zanchettin (2006). This case occurs when

$$\left(1-\tilde{c}_1\right)\left(1-\frac{2-b^2}{b}\right) > \tilde{c}_2-\tilde{c}_1-a \geq \left(1-\tilde{c}_1\right)\left(1-\frac{2}{b}\right) \tag{16}$$

In this case, firm 1 sets a price of $p^I_2(\theta_1,\theta_2)$ where

$$D_1(\tilde{c}_1, p^I_2) = 1-b-ba-\tilde{c}_1+bp^I_2(\theta_1,\theta_2) = 0$$

or

$$p^I_2(\theta_1,\theta_2) = \frac{b+\tilde{c}_1+ba-1}{b}.$$

Note that $p_2^I(\theta_1,\theta_2) < p_2^M(\theta_1,\theta_2) = \frac{1+\tilde{c}_2+a}{2}$ as

$$\begin{aligned}
p_2^I(\theta_1,\theta_2) = \frac{b+\tilde{c}_1+ba-1}{b} &< \frac{1+\tilde{c}_2+a}{2} = p_2^M(\theta_1,\theta_2) \\
2b+2\tilde{c}_1+2ba-2 &< b+b\tilde{c}_2+ab \\
b+2\tilde{c}_1+ba-2 &< b\tilde{c}_2 \\
b+2\tilde{c}_1-b\tilde{c}_1+ba-2 &< b\tilde{c}_2-b\tilde{c}_1 \\
b+2\tilde{c}_1-b\tilde{c}_1-2 &< b\tilde{c}_2-b\tilde{c}_1-ba \\
b+2\tilde{c}_1-b\tilde{c}_1-2 &< b\tilde{c}_2-b\tilde{c}_1-ba \\
(b-2)(1-\tilde{c}_1) &< b\tilde{c}_2-b\tilde{c}_1-ba \\
\left(1-\frac{2}{b}\right)(1-\tilde{c}_1) &< \tilde{c}_2-\tilde{c}_1-a
\end{aligned}$$

where the last inequality holds by inequality (16). Profits in this case are

$$\begin{aligned}
\pi^N_{2,Limit}(\theta_1,\theta_2) &= (1+a-p_2^I)(p_2^I-c_2) \\
&= \left(1+a-\frac{b+\tilde{c}_1+ba-1}{b}\right)\left(\frac{b+\tilde{c}_1+ba-1}{b}-c_2\right) \\
&= \left(\frac{b+ab-(b+\tilde{c}_1+ba-1)}{b}\right)\left(\frac{b+\tilde{c}_1+ba-1}{b}-c_2\right) \\
&= \left(\frac{1-\tilde{c}_1}{b}\right)\left(\frac{b+\tilde{c}_1+ba-1}{b}-c_2\right)
\end{aligned}$$

Derivation of $\pi^N_{2,Monop}(\theta_1,\theta_2)$: When the perceived asymmetry is particularly large, firm 1 is inactive and firm 2 sets the monopoly price. Thus, firm 2's pricing is unconstrained by firm 1. Following Zanchettin (2006), this case occurs when

$$\tilde{c}_2-\tilde{c}_1-a < (1-\tilde{c}_1)\left(1-\frac{2}{b}\right). \tag{17}$$

Note that

$$\begin{aligned}
p_2^I(\theta_1,\theta_2) = \frac{b+\tilde{c}_1+ba-1}{b} &> \frac{1+\tilde{c}_2+a}{2} = p_2^M(\theta_1,\theta_2) \\
2b+2\tilde{c}_1+2ba-2 &> b+b\tilde{c}_2+ab \\
b+2\tilde{c}_1+ba-2 &> b\tilde{c}_2 \\
b+2\tilde{c}_1-b\tilde{c}_1+ba-2 &> b\tilde{c}_2-b\tilde{c}_1 \\
b+2\tilde{c}_1-b\tilde{c}_1-2 &> b\tilde{c}_2-b\tilde{c}_1-ba \\
b+2\tilde{c}_1-b\tilde{c}_1-2 &> b\tilde{c}_2-b\tilde{c}_1-ba \\
(b-2)(1-\tilde{c}_1) &> b\tilde{c}_2-b\tilde{c}_1-ba \\
\left(1-\frac{2}{b}\right)(1-\tilde{c}_1) &> \tilde{c}_2-\tilde{c}_1-a
\end{aligned}$$

where the last inequality holds by inequality (17).

Firm 2's profit in this case is

$$\pi^N_{2,Monop}(\theta_1,\theta_2) = \left(1+a-p_2^M(\theta_1,\theta_2)\right)\left(p_2^M(\theta_1,\theta_2)-c_2\right)$$

where

$$p_2^M(\theta_1,\theta_2) = \frac{1+\tilde{c}_2+a}{2}.$$

Simplifying yields

$$\begin{aligned}
\pi^N_{1,Monop}(\theta_1,\theta_2) &= \left(1+a-\frac{1+\tilde{c}_2+a}{2}\right)\left(\frac{1+\tilde{c}_2+a}{2}-c_2\right)\\
&= \left(\frac{2+2a-1-\tilde{c}_2-a}{2}\right)\left(\frac{1+\tilde{c}_2+a-2c_2}{2}\right)\\
&= \left(\frac{1-\tilde{c}_2+a}{2}\right)\left(\frac{1+c_2\left(1-\theta_2\right)+a-2c_2}{2}\right)\\
&= \left(\frac{1-c_2+c_2\theta_2+a}{2}\right)\left(\frac{1+c_2\left(1-\theta_2\right)+a-2c_2}{2}\right)\\
&= \left(\frac{1-c_2+c_2\theta_2+a}{2}\right)\left(\frac{1-c_2-\theta_2c_2+a}{2}\right)\\
&= \frac{\left(1-c_2+a\right)^2-\left(c_2\theta_2\right)^2}{4}
\end{aligned}$$

Continuity of $\pi^N_2(\theta_1,\theta_2)$: The following lemma establishes that $\pi^N_2(\theta_1,\theta_2)$ is continuous in θ_2.

Lemma A.10. *$\pi^N_2(\theta_1,\theta_2)$ is continuous in θ_2.*

Proof. It suffices to show that

Part 1: $\pi^N_{2,Both}(\theta_1,\theta_2)=0$ when $(1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right)=x-a$

Part 2: $\pi^N_{2,Both}(\theta_1,\theta_2)=\pi^N_{2,Limit}(\theta_1,\theta_2)$ when $(1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right)=x-a$

Part 3: $\pi^N_{2,Limit}(\theta_1,\theta_2)=\pi^N_{2,Monop}(\theta_1,\theta_2)$ when $\left(1-\frac{2}{b}\right)(1-\tilde{c}_1)=x-a$

Part 1: Note that

$$\begin{aligned}
\tilde{c}_2-\tilde{c}_1-a &= (1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right)\\
\tilde{c}_2-\tilde{c}_1-a-(1-\tilde{c}_1) &= \left(-\frac{b}{2-b^2}\right)(1-\tilde{c}_1)\\
\tilde{c}_2-a-1 &= \left(-\frac{b}{2-b^2}\right)(1-\tilde{c}_1)\\
0 &= (-b)\left(1-\tilde{c}_1\right)+\left(1-\tilde{c}_2+a\right)\left(2-b^2\right)\\
0 &= \left(1-\tilde{c}_2+a\right)\left(2-b^2\right)-b\left(1-\tilde{c}_1\right)\\
\implies 0 &= D_2(p^N_1(\theta_1,\theta_2),p^N_2(\theta_1,\theta_2))
\end{aligned}$$

which implies

$$\pi^N_{2,Both}(\theta_1,\theta_2)=0$$

when $(1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right)=x-a$.

Part 2: Note that

$$\begin{aligned}
(1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right) &= \tilde{c}_2-\tilde{c}_1-a\\
(1-\tilde{c}_1)\left(-\frac{2-b^2}{b}\right) &= \tilde{c}_2-\tilde{c}_1-a-1+\tilde{c}_1\\
(1-\tilde{c}_1)\left(-\frac{2-b^2}{b}\right) &= \tilde{c}_2-a-1\\
(1-\tilde{c}_1)\left(2-b^2\right) &= b\left(1-\tilde{c}_2+a\right)\\
\frac{(1-\tilde{c}_1)}{b} &= \frac{(1-\tilde{c}_2+a)}{(2-b^2)}
\end{aligned}$$

Thus, $\frac{(1-\tilde{c}_1)}{b} = \frac{(1-\tilde{c}_2+a)}{(2-b^2)}$ and holds when $(1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right) = \tilde{c}_2 - \tilde{c}_1 - a$.
Substituting $\frac{(1-\tilde{c}_2+a)}{b} = \frac{1}{2-b^2}(1-\tilde{c}_1)$ into $\pi^N_{2,Limit}(\theta_1,\theta_2)$ yields

$$\begin{aligned}
\pi^N_{2,Limit}(\theta_1,\theta_2) &= \left(\frac{1-\tilde{c}_1}{b}\right)\left(\frac{b+\tilde{c}_1+ba-1}{b} - c_2\right) \\
&= \left(\frac{1-\tilde{c}_2+a}{2-b^2}\right)\left(1+a+\frac{\tilde{c}_1-1}{b} - c_2\right) \\
&= \left(\frac{1-\tilde{c}_2+a}{2-b^2}\right)\left(1+a+\frac{1-\tilde{c}_2+a}{2-b^2} - c_2\right) \\
&= \left(\frac{1-\tilde{c}_2+a}{2-b^2}\right)\left(1-\tilde{c}_2+a+\frac{1-\tilde{c}_2+a}{2-b^2} + \tilde{c}_2 - c_2\right) \\
&= \left(\frac{1-\tilde{c}_2+a}{2-b^2}\right)\left((1-\tilde{c}_2+a)\left(1-\frac{1}{2-b^2}\right) + \tilde{c}_2 - c_2\right) \\
&= \left(\frac{1-\tilde{c}_2+a}{2-b^2}\right)\left((1-\tilde{c}_2+a)\frac{1-b^2}{2-b^2} + \tilde{c}_2 - c_2\right)
\end{aligned}$$

Substituting $(1-\tilde{c}_2+a)\frac{b}{2-b^2} = (1-\tilde{c}_1)$ into $\pi^N_{2,Both}(\theta_1,\theta_2)$ yields

$$\begin{aligned}
\pi^N_{2,Both}(\theta_1,\theta_2) &= D_2\left(p^N_1(\theta_1,\theta_2), p^N_2(\theta_1,\theta_2)\right)\left(p^N_2(\theta_1,\theta_2) - c_2\right) \\
&= \frac{1}{1-b^2}\left[\frac{\left(2-b^2\right)(1-\tilde{c}_2+a) - b(1-\tilde{c}_1)}{4-b^2}\right]\left(\frac{\left(2-b^2\right)(1-\tilde{c}_2+a) - b(1-\tilde{c}_1)}{4-b^2} + (\tilde{c}_2 - c_2)\right) \\
&= \frac{1}{1-b^2}\left[\frac{\left(\left(2-b^2\right)(1-\tilde{c}_2+a) - b\left(\frac{b}{2-b^2}(1-\tilde{c}_2+a)\right)\right)}{4-b^2}\right] \\
&\times\left(\frac{\left(2-b^2\right)(1-\tilde{c}_2+a) - b\left(\frac{b}{2-b^2}(1-\tilde{c}_2+a)\right)}{4-b^2} + (\tilde{c}_2 - c_2)\right) \\
&= \frac{1}{(1-b^2)(2-b^2)}\left[\frac{\left(\left(2-b^2\right)^2(1-\tilde{c}_2+a) - b^2(1-\tilde{c}_2+a)\right)}{4-b^2}\right] \\
&\times\left(\frac{1}{2-b^2}\frac{\left(2-b^2\right)^2(1-\tilde{c}_2+a) - b^2(1-\tilde{c}_2+a)}{4-b^2} + (\tilde{c}_2 - c_2)\right) \\
&= \frac{1}{(1-b^2)(2-b^2)}\left[\frac{\left(4-5b^2+b^4\right)(1-\tilde{c}_2+a)}{4-b^2}\right]\left(\frac{1}{2-b^2}\frac{\left(4-5b^2+b^4\right)(1-\tilde{c}_2+a)}{4-b^2} + (\tilde{c}_2 - c_2)\right) \\
&= \frac{1}{(1-b^2)(2-b^2)}\left[\frac{\left(4-b^2\right)\left(1-b^2\right)(1-\tilde{c}_2+a)}{4-b^2}\right]\left(\frac{1}{2-b^2}\frac{\left(4-b^2\right)\left(1-b^2\right)(1-\tilde{c}_2+a)}{4-b^2} + (\tilde{c}_2 - c_2)\right) \\
&= \frac{1}{(2-b^2)}(1-\tilde{c}_2+a)\left(\frac{1}{2-b^2}\left(1-b^2\right)(1-\tilde{c}_2+a) + (\tilde{c}_2 - c_2)\right)
\end{aligned}$$

which implies $\pi^N_{2,Both}(\theta_1,\theta_2) = \pi^N_{2,Limit}(\theta_1,\theta_2)$ when$(1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right) = \tilde{c}_2 - \tilde{c}_1 - a$.

Part 3: It suffices to show that $p_2^I(\theta_1,\theta_2) = p_2^M(\theta_1,\theta_2)$ when $\left(1-\frac{2}{b}\right)(1-\tilde{c}_1) = \tilde{c}_2 - \tilde{c}_1 - a$.

$$p_2^I(\theta_1,\theta_2) = \frac{b+\tilde{c}_1+ba-1}{b} = \frac{1+\tilde{c}_2+a}{2} = p_2^M(\theta_1,\theta_2)$$
$$\iff 2b+2\tilde{c}_1+2ba-2 = b+b\tilde{c}_2+ab$$
$$\iff b+2\tilde{c}_1+ba-2 = b\tilde{c}_2$$
$$\iff b+2\tilde{c}_1-b\tilde{c}_1+ba-2 = b\tilde{c}_2-b\tilde{c}_1$$
$$\iff b+2\tilde{c}_1-b\tilde{c}_1-2 = b\tilde{c}_2-b\tilde{c}_1-ba$$
$$\iff b+2\tilde{c}_1-b\tilde{c}_1-2 = b\tilde{c}_2-b\tilde{c}_1-ba$$
$$\iff (b-2)(1-\tilde{c}_1) = b\tilde{c}_2-b\tilde{c}_1-ba$$
$$\iff \left(1-\frac{2}{b}\right)(1-\tilde{c}_1) = \tilde{c}_2-\tilde{c}_1-a$$

Thus, $\pi_{2,Limit}^N(\theta_1,\theta_2) = \pi_{2,Monop}^N(\theta_1,\theta_2)$ when $\left(1-\frac{2}{b}\right)(1-\tilde{c}_1) = \tilde{c}_2-\tilde{c}_1-a$. □

A.7.2 Two Stage Delegation Game (Manager Competition)

In this subsection, we present the sub-game perfect Nash equilibrium of the two stage delegation game. Formally, we assume that owners simultaneously and publicly select $\theta_i \in [0,1]$ in stage 1. In stage 2, managers compete in prices. There is no possibility of collusion between managers. This is the case when managers discount factors are sufficiently low that collusion is unsustainable for all compensation structures.

We restrict attention to sub-game perfect Nash equilibrium. Additionally, we consider moderate asymmetries between firms. Specifically, we consider asymmetries in marginal cost that satisfy:

$$(1-c_1)\left(1-\frac{b}{2-b^2}\right) > c_2-c_1-a > (1-c_1)\left(1-\frac{2-b^2}{b}\right).$$

When the above inequality is violated (i.e., the asymmetry between firms is sufficiently large), there exists a vast multiplicity of equilibria wherein one firm does not produce in the sub-game perfect equilibrium.

Lemma A.11. *Assume* $(1-c_1)\left(1-\frac{b}{2-b^2}\right) > c_2-c_1-a > (1-c_1)\left(1-\frac{2-b^2}{b}\right)$. *Then,* $\theta_1^*=0$ *and* $\theta_2^*=0$ *in any sub-game perfect Nash equilibrium.*

Proof. First, note that $\pi_1^N(\theta_1^*,\theta_2^*) = \pi_1^N(0,0) = \pi_{1,Both}^N(0,0)$ and $\pi_2^N(\theta_1^*,\theta_2^*) = \pi_2^N(0,0) = \pi_{2,Both}^N(0,0)$ by $(1-c_1)\left(1-\frac{b}{2-b^2}\right) > c_2-c_1-a > (1-c_1)\left(1-\frac{2-b^2}{b}\right)$ (see Subsection A.7.1). First, we show no owner wishes to defect when $\theta_1^*=0$ and $\theta_2^*=0$. The result follows from the fact that $\pi_1^N(\theta_1,0)$ is non-increasing in θ_1 (see the proof of Lemma 3 from the text appendix) and $\pi_2^N(0,\theta_2)$ is non-increasing in θ_2 (Lemma 3 from the text appendix). Thus, owner 1 and owner 2 have no incentive to deviate. Next, we show $\theta_1^*=0$ and $\theta_2^*=0$ in any sub game perfect equilibrium. The result follows from recognizing that $\pi_1^N(\theta_1,\theta_2)$ is strictly decreasing in θ_1 and $\pi_2^N(\theta_1,\theta_2)$ is strictly decreasing in θ_2 when $(1-c_1)\left(1-\frac{b}{2-b^2}\right) > c_2-c_1-a > (1-c_1)\left(1-\frac{2-b^2}{b}\right)$ (see the proof of Lemma 3 in Subsection A.5.7). □

Thus, positive sales weights do not occur in a sub-game perfect equilibrium of the two stage delegation game without collusion.

A.7.3 Two Stage Delegation Game (Manager Collusion)

In this subsection, we analyze a two stage game wherein owners expect managers to collude in the second stage. Formally, owners simultaneously and publicly select their respective manager's sales weight in stage 1. In stage 2, managers set prices to maximize joint pay (i.e., managers collude). We assume owners expect managers to collude in the second stage, regardless of the sales weights chosen in stage 1. This is the case when managers discount factors are sufficiently large that collusion is sustainable for all compensation structures. Additionally, we assume both firms produce during collusion. This occurs when b is sufficiently low (i.e., products are sufficiently differentiated).[10] We restrict attention to sub-game perfect Nash equilibria. Let θ_1^* and θ_2^* denote the sales weights that prevail in a sub-game perfect equilibrium. The following Lemma presents results.

Lemma A.12. *i) If* $b^2(1+a)+2b(1-c_1) < 4c_2$ *and* $b^2+2b(1-c_2+a) < 4c_1$*, then,* $\theta_1^* = \frac{b^2(1-c_1)+2b(1-c_2+a)}{c_1(4-b^2)}$ *and* $\theta_2^* = \frac{b^2(1-c_2+a)+2b(1-c_1)}{c_2(4-b^2)}$.

ii) If $\frac{b}{2c_2} < 1$ *and* $4c_1 \leq 2b(1-c_2+a)+b^2$*, then* $\theta_1^* = 1$ *and* $\theta_2^* = \frac{b}{2c_2}$.

iii) If $\frac{b(1+a)}{2c_1} < 1$ *and* $4c_2 \leq b^2(1+a)+2b(1-c_1)$*, then* $\theta_1^* = \frac{b(1+a)}{2c_1}$ *and* $\theta_2^* = 1$.

iv) If $\frac{b(1+a)}{2c_1} \geq 1$ *and* $\frac{b}{2c_2} \geq 1$*, then* $\theta_1^* = \theta_2^* = 1$.

Proof. Part i) First consider firm 1. Owner 1's profit when managers collude is

$$\begin{aligned}
\pi_1^C(\theta_1,\theta_2) &= D_1\left(p_1^C,p_2^C\right)\left(p_1^C - c_1\right) \\
&= \frac{1}{1-b^2}\left(1-b-ba-\frac{1+\tilde{c}_1}{2}+b\left(\frac{1+\tilde{c}_2+a}{2}\right)\right)\left(\frac{1+\tilde{c}_1}{2}-c_1\right) \\
&= \frac{1}{2(1-b^2)}\left(2-2b-2ba-(1+\tilde{c}_1)+b(1+\tilde{c}_2+a)\right)\left(\frac{1+\tilde{c}_1}{2}-c_1\right) \\
&= \frac{1}{2(1-b^2)}\left(1-\tilde{c}_1-b-ab+b\tilde{c}_2\right)\left(\frac{1+\tilde{c}_1}{2}-c_1\right) \\
&= \frac{1}{2(1-b^2)}\left(1-\tilde{c}_1-b(1-\tilde{c}_2+a)\right)\left(\frac{1+\tilde{c}_1}{2}-c_1\right).
\end{aligned}$$

Owner 1's first order condition is

$$\begin{aligned}
\frac{1}{2(1-b^2)}c_1\left(\frac{1+\tilde{c}_1}{2}-c_1\right)+\frac{1}{2(1-b^2)}\left(1-\tilde{c}_1-b(1-\tilde{c}_2+a)\right)\left(-\frac{c_1}{2}\right) &= 0 \\
\left(\frac{1+\tilde{c}_1}{2}-c_1\right)+\left(1-\tilde{c}_1-b(1-\tilde{c}_2+a)\right)\left(-\frac{1}{2}\right) &= 0 \\
\left(\frac{1+\tilde{c}_1}{2}-c_1\right)+\left(1-\tilde{c}_1-b(1-\tilde{c}_2+a)\right)\left(-\frac{1}{2}\right) &= 0 \\
(1+\tilde{c}_1-2c_1)+\left(1-\tilde{c}_1-b(1-\tilde{c}_2+a)\right)(-1) &= 0 \\
(1+\tilde{c}_1-2c_1)-\left(1-\tilde{c}_1-b(1-\tilde{c}_2+a)\right) &= 0 \\
\tilde{c}_1-2c_1+\tilde{c}_1+b(1-\tilde{c}_2+a) &= 0 \\
2\tilde{c}_1-2c_1+b(1-\tilde{c}_2+a) &= 0 \\
\tilde{c}_1 &= c_1-\frac{b(1-\tilde{c}_2+a)}{2} \\
(1-\theta_1)c_1 &= c_1-\frac{b(1-\tilde{c}_2+a)}{2} \\
-\theta_1 c_1 &= -\frac{b(1-\tilde{c}_2+a)}{2} \\
-\theta_1 &= -\frac{b(1-\tilde{c}_2+a)}{2c_1} \\
\theta_1 &= \frac{b(1-\tilde{c}_2+a)}{2c_1}
\end{aligned}$$

Thus, owner 1's best response function is

$$\theta_1(\theta_2) = \frac{b(1-c_2(1-\theta_2)+a)}{2c_1}. \tag{18}$$

[10] This assumption is made, in part, for consistency with the model in the main text wherein collusion is only be sustainable when both managers produce during collusion.

Next, consider firm 2. Owner 2's profit when managers collude is

$$\begin{aligned}
\pi_2^C(\theta_1,\theta_2) &= D_2\left(p_1^C,p_2^C\right)\left(p_2^C-c_2\right)\\
&= \frac{1}{1-b^2}\left(1-b+a-\frac{1+\tilde{c}_2+a}{2}+b\left(\frac{1+\tilde{c}_1}{2}\right)\right)\left(\frac{1+\tilde{c}_2+a}{2}-c_2\right)\\
&= \frac{1}{2\left(1-b^2\right)}\left(2-2b+2a-\left(1+\tilde{c}_2+a\right)+b\left(1+\tilde{c}_1\right)\right)\left(\frac{1+\tilde{c}_2+a}{2}-c_2\right)\\
&= \frac{1}{2\left(1-b^2\right)}\left(1-\tilde{c}_2-b+a+b\tilde{c}_1\right)\left(\frac{1+\tilde{c}_2+a}{2}-c_2\right)\\
&= \frac{1}{2\left(1-b^2\right)}\left(1-\tilde{c}_2+a-b\left(1-\tilde{c}_1\right)\right)\left(\frac{1+\tilde{c}_2+a}{2}-c_2\right).
\end{aligned}$$

Owner 2's first order condition is

$$\begin{aligned}
\frac{1}{2\left(1-b^2\right)}\left(c_2\right)\left(\frac{1+\tilde{c}_2+a}{2}-c_2\right)+\frac{1}{2\left(1-b^2\right)}\left(1-\tilde{c}_2+a-b\left(1-\tilde{c}_1\right)\right)\left(\frac{-c_2}{2}\right) &= 0\\
\frac{1}{2\left(1-b^2\right)}\left(c_2\right)\left(\frac{1+\tilde{c}_2+a}{2}-c_2\right)+\frac{1}{2\left(1-b^2\right)}\left(1-\tilde{c}_2+a-b\left(1-\tilde{c}_1\right)\right)\left(\frac{-c_2}{2}\right) &= 0\\
\left(1+\tilde{c}_2+a-2c_2\right)-\left(1-\tilde{c}_2+a-b\left(1-\tilde{c}_1\right)\right) &= 0\\
\left(2\tilde{c}_2+a-2c_2\right)-a+b\left(1-\tilde{c}_1\right) &= 0\\
2\tilde{c}_2-2c_2+b\left(1-\tilde{c}_1\right) &= 0\\
2c_2\left(1-\theta_2\right)-2c_2+b\left(1-\tilde{c}_1\right) &= 0\\
-2c_2\theta_2+b\left(1-\tilde{c}_1\right) &= 0\\
\theta_2 &= \frac{b\left(1-\tilde{c}_1\right)}{2c_2}
\end{aligned}$$

Thus, Owner 2's best response function is

$$\theta_2(\theta_1)=\frac{b\left(1-c_1\left(1-\theta_1\right)\right)}{2c_2}. \tag{19}$$

Note that

$$\theta_1\left(\theta_2\right)=\frac{b\left(1-\tilde{c}_2+a\right)}{2c_1}\implies 2\tilde{c}_1-2c_1+b\left(1-\tilde{c}_2+a\right)=0 \tag{20}$$

and

$$\begin{aligned}
\theta_2\left(\theta_1\right)=\frac{b\left(1-\tilde{c}_1\right)}{2c_2} &\implies 2\tilde{c}_2-2c_2+b\left(1-\tilde{c}_1\right)=0\\
&\implies \tilde{c}_2=c_2-\frac{b\left(1-\tilde{c}_1\right)}{2}. \qquad (21)
\end{aligned}$$

Substituting Equation (21) into Equation (20) yields

$$
\begin{aligned}
2\tilde{c}_1 - 2c_1 + b\left(1 - \tilde{c}_2 + a\right) &= 0 \\
2\tilde{c}_1 - 2c_1 + b\left(1 - c_2 + \frac{b\left(1 - \tilde{c}_1\right)}{2} + a\right) &= 0 \\
4\tilde{c}_1 - 4c_1 + b\left(2 - 2c_2 + b\left(1 - \tilde{c}_1\right) + 2a\right) &= 0 \\
4\tilde{c}_1 - 4c_1 + 2b - 2c_2 b + b^2\left(1 - \tilde{c}_1\right) + 2ab &= 0 \\
4\tilde{c}_1 - 4c_1 + 2b - 2c_2 b + b^2 - b^2\tilde{c}_1 + 2ab &= 0
\end{aligned}
$$

$$
\begin{aligned}
\left(4 - b^2\right)\tilde{c}_1 &= 4c_1 - 2b + 2c_2 b - b^2 - 2ab \\
\left(4 - b^2\right)\tilde{c}_1 &= 4c_1 - b^2 - 2b\left(1 - c_2 + a\right) \\
\tilde{c}_1 &= \frac{4c_1 - b^2 - 2b\left(1 - c_2 + a\right)}{\left(4 - b^2\right)} \\
1 - \theta_1 &= \frac{4c_1 - b^2 - 2b\left(1 - c_2 + a\right)}{c_1\left(4 - b^2\right)} \\
\theta_1 &= 1 - \frac{4c_1 - b^2 - 2b\left(1 - c_2 + a\right)}{c_1\left(4 - b^2\right)} \\
\theta_1 &= \frac{4c_1 - b^2 c_1 - 4c_1 + b^2 + 2b\left(1 - c_2 + a\right)}{c_1\left(4 - b^2\right)} \\
\theta_1 &= \frac{b^2\left(1 - c_1\right) + 2b\left(1 - c_2 + a\right)}{c_1\left(4 - b^2\right)}.
\end{aligned}
$$

Thus, $\theta_1^* = \frac{b^2(1-c_1)+2b(1-c_2+a)}{c_1(4-b^2)}$. Analogous derivations show that $\theta_2^* = \frac{b^2(1-c_2+a)+2b(1-c_1)}{c_2(4-b^2)}$.

$\theta_1^* > 0$ as

$$
\begin{aligned}
\theta_1^* = \frac{b^2\left(1 - c_1\right) + 2b\left(1 - c_2 + a\right)}{c_1\left(4 - b^2\right)} &> 0 \\
\iff b\left(1 - c_1\right) + 2\left(1 - c_2 + a\right) &> 0
\end{aligned}
$$

which always holds. $\theta_1^* < 1$ as

$$
\begin{aligned}
\theta_1^* = \frac{b^2\left(1 - c_1\right) + 2b\left(1 - c_2 + a\right)}{c_1\left(4 - b^2\right)} &< 1 \\
b^2\left(1 - c_1\right) + 2b\left(1 - c_2 + a\right) &< c_1\left(4 - b^2\right) \\
b^2 + 2b\left(1 - c_2 + a\right) &< 4c_1
\end{aligned}
$$

which holds by assumption. $\theta_2^* > 0$ as

$$
\begin{aligned}
\frac{b^2\left(1 - c_2 + a\right) + 2b\left(1 - c_1\right)}{c_2\left(4 - b^2\right)} &> 0 \\
\iff b^2\left(1 - c_2 + a\right) + 2b\left(1 - c_1\right) &> 0
\end{aligned}
$$

which always holds. $\theta_2^* < 1$ as

$$
\begin{aligned}
\theta_2^* = \frac{b^2\left(1 - c_2 + a\right) + 2b\left(1 - c_1\right)}{c_2\left(4 - b^2\right)} &< 1 \\
b^2\left(1 - c_2 + a\right) + 2b\left(1 - c_1\right) &< c_2\left(4 - b^2\right) \\
b^2\left(1 + a\right) + 2b\left(1 - c_1\right) &< 4c_2
\end{aligned}
$$

which holds by assumption.

Part ii) When $\theta_1^* = 1$, owner 2's best response is

$$\theta_2^* = \frac{b\left(1 - \tilde{c}_1\right)}{2c_2} = \frac{b}{2c_2}$$

if $\frac{b}{2c_2} < 1$ which holds by assumption. $\theta_1^* = 1$ is owner 1's best response to $\theta_2^* = \frac{b}{2c_2}$ if

$$\theta_1(\theta_2^*) = \frac{b\left(1 - c_2\left(1 - \theta_2^*\right) + a\right)}{2c_1} \geq 1.$$

Substituting

$$c_2\left(1 - \theta_2^*\right) = \left(1 - \frac{b}{2c_2}\right)c_2 = c_2 - \frac{b}{2}$$

into $\frac{b(1-c_2(1-\theta_2^*)+a)}{2c_1} \geq 1$ yields

$$\begin{aligned} \frac{b\left(1 - \left(c_2 - \frac{b}{2}\right) + a\right)}{2c_1} &\geq 1 \\ \iff b\left(1 - \left(c_2 - \frac{b}{2}\right) + a\right) &\geq 2c_1 \\ \iff b\left(1 - c_2 + \frac{b}{2} + a\right) &\geq 2c_1 \\ \iff b\left(1 - c_2 + a\right) + \frac{b^2}{2} &\geq 2c_1 \\ \iff 2b\left(1 - c_2 + a\right) + b^2 &\geq 4c_1 \end{aligned}$$

which holds by assumption.

Part iii) When $\theta_2^* = 1$, owner 1's best response is

$$\theta_1^* = \frac{b\left(1 - \tilde{c}_2 + a\right)}{2c_1} = \frac{b\left(1 + a\right)}{2c_1}$$

if

$$\theta_1^* = \frac{b\left(1 + a\right)}{2c_1} < 1$$

which holds by assumption. Owner 2's best response to θ_1^* is $\theta_2^* = 1$ if

$$\theta_2(\theta_1^*) = \frac{b\left(1 - c_1\left(1 - \theta_1^*\right)\right)}{2c_2} \geq 1.$$

Substituting

$$c_1\left(1 - \theta_1^*\right) = \left(1 - \frac{b\left(1 + a\right)}{2c_1}\right)c_1 = c_1 - \frac{b\left(1 + a\right)}{2}$$

into $\frac{b(1-c_1(1-\theta_1^*))}{2c_2} \geq 1$ yields

$$\begin{aligned} \frac{b\left(1 - \left(c_1 - \frac{b(1+a)}{2}\right)\right)}{2c_2} &\geq 1 \\ \iff b\left(1 - \left(c_1 - \frac{b\left(1 + a\right)}{2}\right)\right) &\geq 2c_2 \\ \iff b\left(1 - c_1 + \frac{b\left(1 + a\right)}{2}\right) &\geq 2c_2 \\ \iff b\left(1 - c_1\right) + b^2\frac{\left(1 + a\right)}{2} &\geq 2c_2 \end{aligned}$$

$$\iff 2b\,(1-c_1)+b^2\,(1+a)\geq 4c_2$$

which holds by assumption.

Part iv) $\theta_1^*=1$ is a best response to θ_2^* if

$$\theta_1(1)=\frac{b\,(1+a)}{2c_1}\geq 1$$

which holds by assumption. $\theta_2^*=1$ is a best response to θ_1^* if

$$\theta_2(1)=\frac{b}{2c_2}\geq 1$$

which holds by assumption. □

A.8 Nash Bargaining

In this section, we explore the robustness of our results to the alternative assumption that collusive prices are determined by Nash bargaining between managers, rather than joint profit maximization. We follow prior literature (Schmalensee, 1987) and analyze outcomes under Nash bargaining numerically (using MATLAB), as analytical solutions are intractable.

For each (θ_1,θ_2) combination, we obtain Nash equilibrium payoffs, a Nash bargaining solution (i.e., collusive payoffs), and defection payoffs. First, the Nash equilibrium profits are

$$M_1^N=\frac{((2-b^2)(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{(4-b^2)^2(1-b^2)},\ \text{and}$$
$$M_2^N=\frac{((2-b^2)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{(4-b^2)^2(1-b^2)}$$

where $\tilde{c}_i=(1-\theta_i)c_i$.

The Nash bargaining solution is a pair of prices p_1^{NB} and p_2^{NB} that maximize the following Nash product $\mathcal{M}$. Each manager earns their Nash equilibrium payoff at the disagreement point.

$$\begin{aligned}\mathcal{M}&=(M_1(p_1,p_2)-M_1^N)(M_2(p_1,p_2)-M_2^N)\\&=(D_1(p_1,p_2)(p_1-\tilde{c}_1)-M_1^N)(D_2(p_1,p_2)(p_2-\tilde{c}_2)-M_2^N)\end{aligned}$$

where M_1^N and M_2^N are Nash equilibrium payoffs. The first-order conditions are

$$\begin{aligned}0&=\frac{\partial\mathcal{M}}{\partial p_1}\\&=\left(\frac{\partial}{\partial p_1}(D_1(p_1,p_2)(p_1-\tilde{c}_1)-M_1^N)\right)\cdot(D_2(p_1,p_2)(p_2-\tilde{c}_2)-M_2^N)\\&+\left(\frac{\partial}{\partial p_1}(D_2(p_1,p_2)(p_2-\tilde{c}_2))-M_2^N)\right)\cdot(D_1(p_1,p_2)(p_1-\tilde{c}_1))-M_1^N)\\&=\left(\frac{\partial}{\partial p_1}D_1(p_1,p_2)\cdot(p_1-\tilde{c}_1)+D_1(p_1,p_2)\right)\cdot(D_2(p_1,p_2)(p_2-\tilde{c}_2))-M_2^N)\\&+\left(\frac{\partial}{\partial p_1}D_2(p_1,p_2)\cdot(p_2-\tilde{c}_2))\right)\cdot(D_1(p_1,p_2)(p_1-\tilde{c}_1)-M_1^N)\\&=\left(-\frac{1}{1-b^2}\cdot(p_1-\tilde{c}_1)+D_1(p_1,p_2)\right)\cdot(D_2(p_1,p_2)(p_2-\tilde{c}_2))-M_2^N)\\&+\frac{b}{1-b^2}\cdot(p_2-\tilde{c}_2)\cdot(D_1(p_1,p_2)(p_1-\tilde{c}_1)-M_1^N)\end{aligned}$$

and

$$\begin{aligned}
0 &= \frac{\partial \mathcal{M}}{\partial p_2} \\
&= \left(\frac{\partial}{\partial p_2}(D_1(p_1,p_2)(p_1-\tilde{c}_1)-M_1^N)\right)\cdot(D_2(p_1,p_2)(p_2-\tilde{c}_2)-M_2^N) \\
&+ \left(\frac{\partial}{\partial p_2}(D_2(p_1,p_2)(p_2-\tilde{c}_2)-M_2^N)\right)\cdot(D_1(p_1,p_2)(p_1-\tilde{c}_1)-M_1^N) \\
&= \left(\frac{\partial}{\partial p_2}D_1(p_1,p_2)\cdot(p_1-\tilde{c}_1)\right)\cdot(D_2(p_1,p_2)(p_2-\tilde{c}_2)-M_2^N) \\
&+ \left(\frac{\partial}{\partial p_2}D_2(p_1,p_2)\cdot(p_2-\tilde{c}_2)+D_2(p_1,p_2)\right)\cdot(D_1(p_1,p_2)(p_1-\tilde{c}_1)-M_1^N) \\
&= \left(\frac{b}{1-b^2}\cdot(p_1-\tilde{c}_1)\right)\cdot(D_2(p_1,p_2)(p_2-\tilde{c}_2)-M_2^N) \\
&+ \left(D_2(p_1,p_2)-\frac{1}{1-b^2}\cdot(p_2-\tilde{c}_2)\right)\cdot(D_1(p_1,p_2)(p_1-\tilde{c}_1)-M_1^N).
\end{aligned}$$

Thus, the Nash Bargaining solution, denoted (p_1^{NB}, p_2^{NB}), solves the following system of equations:

$$\begin{cases}
0 & = \big((1-b^2)D_1(p_1,p_2)-(p_1-\tilde{c}_1)\big)\cdot\big(D_2(p_1,p_2)(p_2-\tilde{c}_2)-M_2^N\big) \\
& +b\cdot(p_2-\tilde{c}_2)\cdot\big(D_1(p_1,p_2)(p_1-\tilde{c}_1)-M_1^N\big) \\
0 & = b\cdot(p_1-\tilde{c}_1)\cdot\big(D_2(p_1,p_2)(p_2-\tilde{c}_2)-M_2^N\big) \\
& +\big((1-b^2)D_2(p_1,p_2)-(p_2-\tilde{c}_2)\big)\cdot\big(D_1(p_1,p_2)(p_1-\tilde{c}_1)-M_1^N\big).
\end{cases}$$

When a manager defects from the collusive agreement, they set a price which is a best response to the the other manager's Nash bargaining price. Manager 1 solves the following problem.

$$\max_p M_1(p, p_2^{NB}) = D_1(p, p_2^{NB})\cdot(p-\tilde{c}_1)$$

Note that the demand for firm 1 is

$$D_1(p_1,p_2) = \begin{cases}
1-p_1 & \text{if } \; 1-b+a+b\cdot p_1 < p_2 \\
& \;\text{and } p_1 < 1 \\
\frac{1}{1-b^2}(1-b-a\cdot b-p_1+b\cdot p_2) & \text{if } \; 1-b+a+b\cdot p_1 > p_2 \\
& \;\text{and } 1-b-a\cdot b+b\cdot p_2 > p_1 \\
0 & \text{if } \; 1-b-a\cdot b+b\cdot p_2 < p_1
\end{cases}.$$

The derivative of firm 1's demand with respect to price is

$$\frac{\partial D_1(p,p_2)}{\partial p} = \begin{cases}
-1 & \text{if } \; 1-b+a+b\cdot p < p_2 \\
& \;\text{and } p < 1 \\
-\frac{1}{1-b^2} & \text{if } \; 1-b+a+b\cdot p > p_2 \\
& \;\text{and } 1-b-a\cdot b+b\cdot p_2 > p \\
0 & \text{if } \; 1-b-a\cdot b+b\cdot p_2 < p
\end{cases}.$$

Thus,

$$\frac{\partial M_1(p,p_2^{NB})}{\partial p} = D_1(p,p_2^{NB}) + \frac{\partial D_1(p,p_2^{NB})}{\partial p}\cdot(p-\tilde{c}_1)$$

$$= \begin{cases} 1-p-(p-\tilde{c}_1) & \text{if} \quad 1-b+a+b\cdot p < p_2^{NB} \\ & \text{ and } p<1 \\ \frac{1}{1-b^2}(1-b-a\cdot b-p+b\cdot p_2^{NB}) - \frac{(p-\tilde{c}_1)}{1-b^2} & \text{if} \quad 1-b+a+b\cdot p > p_2^{NB} \\ & \text{ and } 1-b-a\cdot b+b\cdot p_2^{NB} > p \\ 0 & \text{if} \quad 1-b-a\cdot b+b\cdot p_2^{NB} < p \end{cases}$$

$$= \begin{cases} 1+\tilde{c}_1-2p & \text{if} \quad 1-b+a+b\cdot p < p_2^{NB} \\ & \text{ and } p<1 \\ \frac{1}{1-b^2}\cdot(1+\tilde{c}_1-b-ab+bp_2^{NB}-2p) & \text{if} \quad 1-b+a+b\cdot p > p_2^{NB} \\ & \text{ and } 1-b-a\cdot b+b\cdot p_2^{NB} > p \\ 0 & \text{if} \quad 1-b-a\cdot b+b\cdot p_2^{NB} < p \end{cases}$$

and, finally, the optimal defection price for manager 1, p_1^D is

$$p_1^D = \begin{cases} \frac{1+\tilde{c}_1}{2} & \text{if} \quad \frac{1+\tilde{c}_1}{2} < \frac{p_2^{NB}-(1-b+a)}{b} \\ \frac{p_2^{NB}-(1-b+a)}{b} & \text{if} \quad \frac{1+\tilde{c}_1}{2} - b\cdot\frac{1+a-p_2^{NB}}{2} < \frac{p_2^{NB}-(1-b+a)}{b} < \frac{1+\tilde{c}_1}{2} \\ \frac{1+\tilde{c}_1}{2} - b\cdot\frac{1+a-p_2^{NB}}{2} & \text{if} \quad \frac{p_2^{NB}-(1-b+a)}{b} < \frac{1+\tilde{c}_1}{2} - b\cdot\frac{1+a-p_2^{NB}}{2} \end{cases}$$

Analogously, manager 2 solves

$$\max_p M_2(p_1^{NB},p) = D_2(p_1^{NB},p)\cdot(p-\tilde{c}_2)$$

where

$$D_2(p_1,p_2) = \begin{cases} 1+a-p_2 & \text{if} \quad 1-b-a\cdot b+b\cdot p_2 < p_1 \\ & \text{and } p_2 < 1+a \\ \frac{1}{1-b^2}(1-b+a-p_2+b\cdot p_1) & \text{if} \quad 1-b-a\cdot b+b\cdot p_2 > p_1 \\ & \text{and} \quad 1-b+a+b\cdot p_1 > p_2 \\ 0 & \text{if} \quad 1-b+a+b\cdot p_1 < p_2 \end{cases}.$$

The derivative of demand for firm 2 with respect to its price is

$$\frac{\partial D_2(p_1,p)}{\partial p} = \begin{cases} -1 & \text{if} \quad 1-b-a\cdot b+b\cdot p < p_1 \\ & \text{and } p < 1+a \\ -\frac{1}{1-b^2} & \text{if} \quad 1-b-a\cdot b+b\cdot p > p_1 \\ & \text{and} \quad 1-b+a+b\cdot p_1 > p \\ 0 & \text{if} \quad 1-b+a+b\cdot p_1 < p \end{cases}$$

Thus,

$$\frac{\partial M_2(p_1^{NB}, p)}{\partial p} = D_2(p_1^{NB}, p) + \frac{\partial D_2(p_1^{NB}, p)}{\partial p} \cdot (p - \tilde{c}_2)$$

$$= \begin{cases} 1 + a - p - (p - \tilde{c}_2) & \text{if} \quad 1 - b - a \cdot b + b \cdot p < p_1^{NB} \\ & \text{and } p < 1 + a \\ \frac{1}{1-b^2}(1 - b + a - p + b \cdot p_1^{NB}) - \frac{p - \tilde{c}_2}{1-b^2} & \text{if} \quad 1 - b - a \cdot b + b \cdot p > p_1^{NB} \\ & \text{and} \quad 1 - b + a + b \cdot p_1^{NB} > p \\ 0 & \text{if} \quad 1 - b + a + b \cdot p_1^{NB} < p \end{cases}$$

$$= \begin{cases} 1 + a + \tilde{c}_2 - 2p & \text{if} \quad 1 - b - a \cdot b + b \cdot p < p_1^{NB} \\ & \text{and } p < 1 + a \\ \frac{1}{1-b^2}(1 + a + \tilde{c}_2 - b + b \cdot p_1^{NB} - 2p) & \text{if} \quad 1 - b - a \cdot b + b \cdot p > p_1^{NB} \\ & \text{and} \quad 1 - b + a + b \cdot p_1^{NB} > p \\ 0 & \text{if} \quad 1 - b + a + b \cdot p_1^{NB} < p \end{cases}$$

and, finally, the optimal defection price for manager 2, p_2^D is

$$p_2^D = \begin{cases} \frac{1+a+\tilde{c}_2}{2} - b \cdot \frac{1-p_1^{NB}}{2} & \text{if} \quad \frac{p_1^{NB} - (1-b-ab)}{b} < \frac{1+a+\tilde{c}_2}{2} - b \cdot \frac{1-p_1^{NB}}{2} \\ \frac{p_1^{NB} - (1-b-ab)}{b} & \text{if} \quad \frac{1+a+\tilde{c}_2}{2} - b \cdot \frac{1-p_1^{NB}}{2} < \frac{p_1^{NB} - (1-b-ab)}{b} < \frac{1+a+\tilde{c}_2}{2} \\ \frac{1+a+\tilde{c}_2}{2} & \text{if} \quad \frac{1+a+\tilde{c}_2}{2} < \frac{p_1^{NB} - (1-b-ab)}{b} \end{cases}$$

Next, we present numerical results. Figure 1 and 2 depict the set of (θ_1, θ_2) values that satisfy the condition $\delta^*(\theta_1, \theta_2) < \delta^*(0, 0)$. These figures correspond to Figure 1 and 2 in the main article and depict (θ_1, θ_2) pairs that facilitate collusion when managers engage in Nash bargaining during collusion. Note that the set of (θ_1, θ_2) pairs that facilitate collusion under Nash bargaining closely resembles the set of (θ_1, θ_2) pairs that facilitate collusion under joint profit maximization (see Figure 1 and 2 in the main text). Specifically, sales weights that reduce the perceived asymmetry between firms tend to facilitate collusion. This suggests that the results of Section 4 may also hold under Nash bargaining.

Figure 3 depicts the set of (θ_1, θ_2) values for which 1) manager collusion is sustainable and 2) the condition $\pi_i^C(\theta_1, \theta_2) > \pi_i^N(0, 0)$ is satisfied for $i = 1, 2$ (i.e., profits under sales-based compensation and manager collusion exceed profits under profit-based compensation and manager competition). Results suggest that sales-based compensation can potentially enhance both firms' profits. The set of (θ_1, θ_2) pairs in Figure 3 closely resembles the corresponding set in the main text (see Figure 3 in the main text).

Figure 4 displays the set of (θ_1, θ_2) values for which 1) manager collusion is sustainable and 2) the condition $CS^C(\theta_1, \theta_2) < CS^N(0, 0)$ is satisfied. Results indicate that sales-based compensation may reduce consumer welfare. Note that, as under profit maximization (see Condition 3a and Figure 4 in the main text), sales-based compensation reduces consumer surplus for sufficiently small values of θ_1 and/or θ_2. When sales weights are large, the price-distorting effect dominates and sales-based compensation can reduce prices and enhance consumer surplus.

Finally, Figure 5 portrays the (θ_1, θ_2) values that are collusion-compatible (i.e., the incentive compatibility constraints of owners are satisfied and collusion between managers is sustainable) when managers engage in Nash bargaining during collusion. Overall, the results of this section suggest that the main qualitative findings in the main text can also hold when managers engage in Nash bargaining during collusion (rather than joint profit maximization).

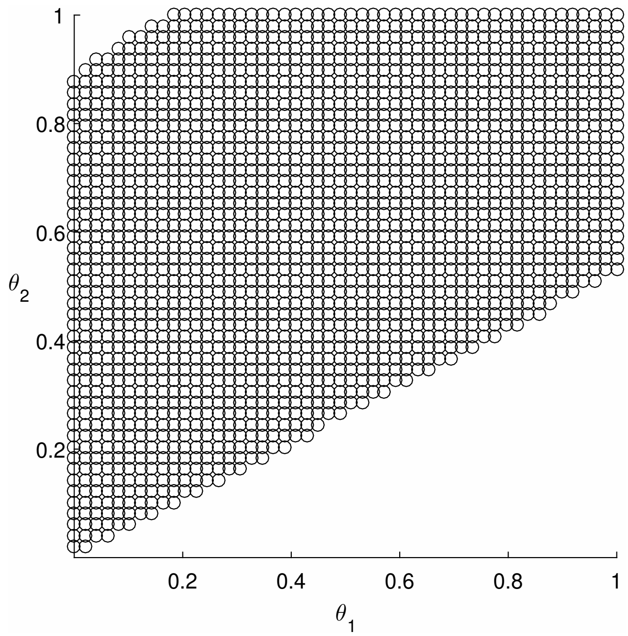

Figure 1: (θ_1, θ_2) values for which Sales-based Compensation Facilitates Collusion under Nash Bargaining when $a = 0$, $b = 0.2$, $c_1 = 0.1$, and $c_2 = 0.175$

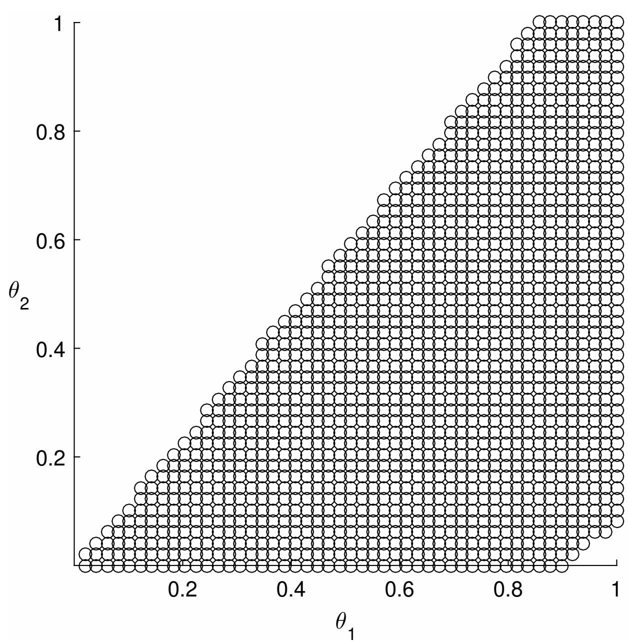

Figure 2: (θ_1, θ_2) values for which Sales-based Compensation Facilitates Collusion under Nash Bargaining when $a = 0.125$, $b = 0.2$, and $c_1 = c_2 = 0.3$

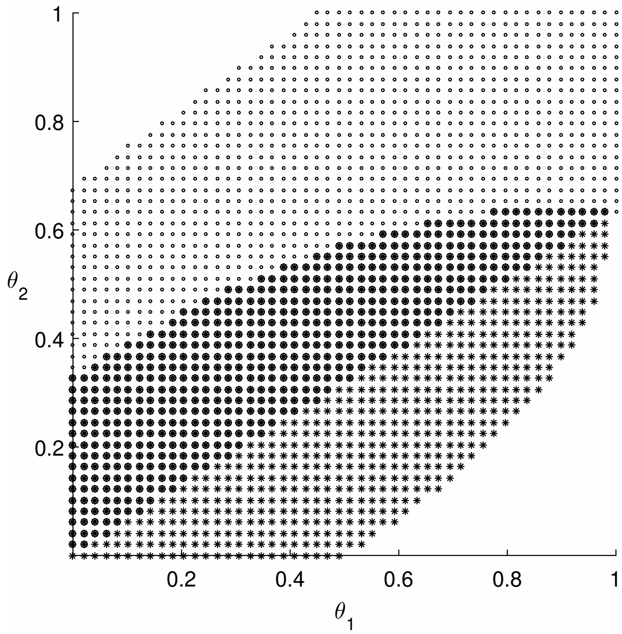

Figure 3: (θ_1, θ_2) values for which Sales-based Compensation Enhances Owner Profit under Nash Bargaining when $a = 0$, $b = 0.2$, $c_1 = 0.1$, $c_2 = 0.15$, and $\delta = 0.5039$. The light gray region delineates θ_1 and θ_2 values for which collusion is sustainable. The dark gray region delineates θ_1 and θ_2 values satisfying $\pi_1^C(\theta_1, \theta_2) > \pi_1^N(0, 0)$ and $\pi_2^C(\theta_1, \theta_2) > \pi_2^N(0, 0)$.

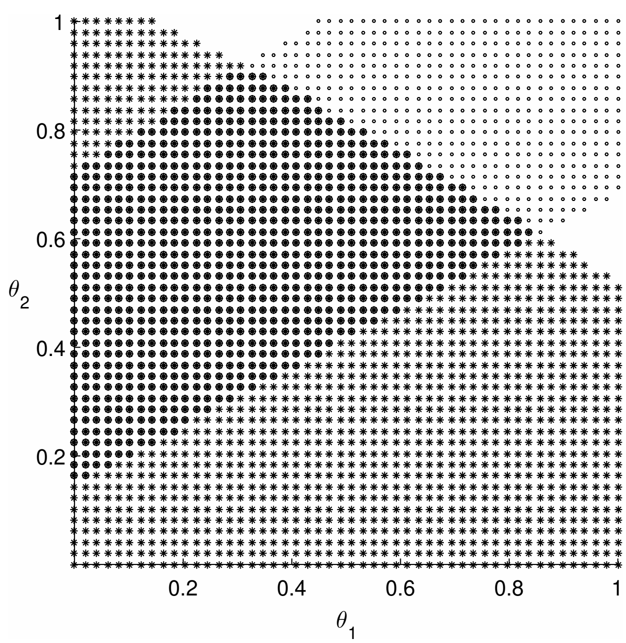

Figure 4: (θ_1, θ_2) values for which Sales-based Compensation Reduces Consumer Surplus under Nash Bargaining when $a = 0$, $b = 0.2$, $c_1 = 0.1$, $c_2 = 0.175$, and $\delta = 0.5039$. The light gray region delineates θ_1 and θ_2 values for which collusion is sustainable. The dark gray region delineates θ_1 and θ_2 values satisfying $CS^C(\theta_1, \theta_2) < CS^N(0, 0)$.

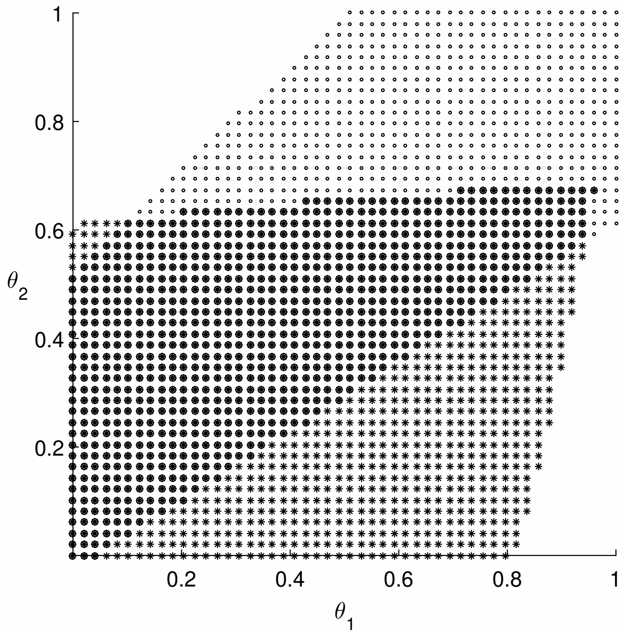

Figure 5: Collusion-compatible (θ_1, θ_2) values under Nash Bargaining when $a = 0$, $b = 0.375$, $c_1 = 0.35$, $c_2 = 0.45$, and $\delta = 0.52$. The light gray region delineates θ_1 and θ_2 values for which collusion is sustainable for managers. The dark gray region delineates θ_1 and θ_2 values for which no owner wishes to defect in the initial stage.

B Quantity Competition

In this section, we present results under differentiated product Cournot competition.

B.1 Preliminaries

The following assumption (which mirrors the assumption in Section 2 of the main text) ensures the critical discount factor is less than 1 under differentiated product Cournot competition (see Lemma B.5 for a proof). If this assumption does not hold, collusion is unsustainable for any discount factor. We assume the following for the remainder of Section B. As under price competition, we assume $\tilde{c}_2 < 1 + a$ and $\tilde{c}_1 < 1$. Recall that x is defined as the asymmetry in perceived marginal cost, $x \equiv \tilde{c}_2 - \tilde{c}_1$.

Assumption 2. *$x - a \in (y_L, y_U)$ where*

$$y_L = (1 - \tilde{c}_1)\left(1 + \frac{\left(8b + b^2\sqrt{b^2+8} + b^3 - 4\sqrt{b^2+8}\right)}{8\left(1-b^2\right)}\right)$$

and

$$y_U = (1 - \tilde{c}_1)\left(1 + \frac{\left(-8b - b^3 + b^2\sqrt{b^2+8} - 4\sqrt{b^2+8}\right)}{2\left(b^2+8\right)}\right).$$

The following lemma will be used throughout Section B.

Lemma B.1. *i) $y_L < x - a < y_U \implies -(1-\tilde{c}_1)(1-b) < x - a < (1-\tilde{c}_1)(1-b)$,*
ii) $y_L < x - a < y_U \implies (1-\tilde{c}_1)\left(1 - \frac{2-b^2}{b}\right) < x - a < (1-\tilde{c}_1)\left(1 - \frac{b}{2-b^2}\right)$, and
iii) $y_L < x - a < y_U \implies (1-\tilde{c}_1)\left(1 - \frac{1}{b}\right) < x - a < (1-\tilde{c}_1)(1-b)$.

Proof. Part i) The proof follows from[11]

$$\begin{aligned}
& y_U = (1 - \tilde{c}_1)\left(1 + \frac{\left(-8b - b^3 + b^2\sqrt{b^2+8} - 4\sqrt{b^2+8}\right)}{2\left(b^2+8\right)}\right) < (1-\tilde{c}_1)(1-b) \\
\iff\ & 1 + \frac{\left(-8b - b^3 + b^2\sqrt{b^2+8} - 4\sqrt{b^2+8}\right)}{2\left(b^2+8\right)} < 1 - b \\
\iff\ & \frac{-8b - b^3 + b^2\sqrt{b^2+8} - 4\sqrt{b^2+8}}{2\left(b^2+8\right)} < -b \\
\iff\ & b < \frac{8b + b^3 - b^2\sqrt{b^2+8} + 4\sqrt{b^2+8}}{2\left(b^2+8\right)} \\
\iff\ & 16b + 2b^3 < 8b + b^3 - b^2\sqrt{b^2+8} + 4\sqrt{b^2+8} \\
\iff\ & 16b + 2b^3 < \left(4 - b^2\right)\sqrt{b^2+8} + 8b + b^3 \\
& 8b + b^3 < \left(4 - b^2\right)\sqrt{b^2+8}
\end{aligned}$$

[11]Note that $\tilde{c}_1 \le c_1 < 1$ where the last inequality follows by Assumption.

which holds for all $b > 0$, and

$$
\begin{aligned}
& y_L = (1-\tilde{c}_1)\left(1+\frac{\left(8b+b^2\sqrt{b^2+8}+b^3-4\sqrt{b^2+8}\right)}{8\left(1-b^2\right)}\right) > -\left(1-\tilde{c}_1\right)\left(1-b\right) \\
\iff & 1+\frac{\left(8b+b^2\sqrt{b^2+8}+b^3-4\sqrt{b^2+8}\right)}{8\left(1-b^2\right)} > -1+b \\
\iff & \frac{\left(8b+b^2\sqrt{b^2+8}+b^3-4\sqrt{b^2+8}\right)}{8\left(1-b^2\right)} > -2+b \\
\iff & 2-b > -\frac{\left(8b+b^2\sqrt{b^2+8}+b^3-4\sqrt{b^2+8}\right)}{8\left(1-b^2\right)} \\
\iff & 2-b > \frac{-8b-b^3+\left(4-b^2\right)\sqrt{b^2+8}}{8\left(1-b^2\right)} \\
& \left(8\left(1-b^2\right)\right)\left(2-b\right) > -8b-b^3+\left(4-b^2\right)\sqrt{b^2+8}
\end{aligned}
$$

which holds for all $b > 0$.

Part ii) The proof follows from Part i,

$$
\begin{aligned}
& (1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right) < -(1-\tilde{c}_1)(1-b) \\
\iff & 1-\frac{2-b^2}{b} < -1+b \\
\iff & 2-\frac{2-b^2}{b} < b \\
\iff & 2-b < \frac{2-b^2}{b} \\
\iff & 2b-b^2 < 2-b^2 \\
\iff & 2b < 2,
\end{aligned}
$$

and

$$
\begin{aligned}
& (1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right) > (1-\tilde{c}_1)(1-b) \\
\iff & b > \frac{b}{2-b^2} \\
\iff & 2-b^2 > 1 \\
\iff & 1 > b^2.
\end{aligned}
$$

Part iii) The proof follows from Part i and

$$
\begin{aligned}
& (1-\tilde{c}_1)\left(1-\frac{1}{b}\right) < -(1-\tilde{c}_1)(1-b) \\
\iff & 1-\frac{1}{b} < -(1-b) \\
\iff & 1-\frac{1}{b} < -1+b \\
\iff & 2-\frac{1}{b} < b \\
\iff & 2b-1 < b^2 \\
\iff & 0 < b^2-2b+1 \\
\iff & 0 < (b-1)^2.
\end{aligned}
$$

□

Equivalence of Contract Types The equivalence of contract types shown in Subsection A.1 holds, by the same derivations, under quantity competition.

Additional Technical Lemmas The following technical lemmas will be used to derive critical discount factors and prove Lemma 1 and Lemma 2.

Lemma B.2. $A_2(\tilde{c}_1, x, a) = \frac{b^2((1-\tilde{c}_1)(1-b)+bx)^2}{((2-b-b^2)(1-\tilde{c}_1)-x(2-b^2))^2 - \frac{16(1-b^2)^2}{(b^2-4)^2}((2-b)(1-\tilde{c}_1)-2x)^2}$ *is increasing in* $x - a$.

Proof. Note that $A_2(\tilde{c}_1, x, a) = A_2(\tilde{c}_1, y, 0)$ where y is defined as $y = x - a$. The remainder of the proof, for this case, shows that $\frac{\partial}{\partial y} A_2(\tilde{c}_1, y, 0) > 0$ for y such that $y \in (y_L, y_U)$. $\frac{\partial}{\partial y} A_2(\tilde{c}_1, y, 0) > 0$ when

$$\left(\left((2-b-b^2)(1-\tilde{c}_1) - y(2-b^2)\right)^2 - 16\left(1-b^2\right)^2 \frac{((2-b)(1-\tilde{c}_1)-2y)^2}{(b^2-4)^2}\right) 2b^3\left((1-\tilde{c}_1)(1-b)+by\right)$$
$$-b^2\left((1-\tilde{c}_1)(1-b)+by\right)^2\left(-\left(2-b^2\right)2\left((2-b-b^2)(1-\tilde{c}_1)-y(2-b^2)\right)+64\left(1-b^2\right)^2\frac{((2-b)(1-\tilde{c}_1)-2y)}{(b^2-4)^2}\right) > 0$$

$$\left(\left((2-b-b^2)(1-\tilde{c}_1) - y(2-b^2)\right)^2 - 16\left(1-b^2\right)^2 \frac{((2-b)(1-\tilde{c}_1)-2y)^2}{(b^2-4)^2}\right) b$$
$$-\left((1-\tilde{c}_1)(1-b)+by\right)\left(-\left(2-b^2\right)\left((2-b-b^2)(1-\tilde{c}_1)-y(2-b^2)\right)+32\left(1-b^2\right)^2\frac{((2-b)(1-\tilde{c}_1)-2y)}{(b^2-4)^2}\right) > 0$$

$$\left(\left((2-b-b^2)(1-\tilde{c}_1)\right)^2 - 2y\left(2-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1) + y^2\left(2-b^2\right)^2\right) b$$
$$+\left(-16\frac{\left(1-b^2\right)^2}{(b^2-4)^2}\left(((2-b)(1-\tilde{c}_1))^2 - 4(2-b)(1-\tilde{c}_1)y + 4y^2\right)\right) b$$
$$-\left((1-\tilde{c}_1)(1-b)+by\right)\left(-\left(2-b^2\right)\left((2-b-b^2)(1-\tilde{c}_1)-y(2-b^2)\right)+32\left(1-b^2\right)^2\frac{((2-b)(1-\tilde{c}_1)-2y)}{(b^2-4)^2}\right) > 0$$

$$\left((2-b-b^2)(1-\tilde{c}_1)\right)^2 b - 2y\left(2-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1) b + y^2\left(2-b^2\right)^2 b$$
$$-16b\frac{\left(1-b^2\right)^2}{(b^2-4)^2}((2-b)(1-\tilde{c}_1))^2 + 64b\frac{\left(1-b^2\right)^2}{(b^2-4)^2}(2-b)(1-\tilde{c}_1)y - 16b\frac{\left(1-b^2\right)^2}{(b^2-4)^2}4y^2$$
$$+(1-b)\left(2-b^2\right)\left(\left(2-b-b^2\right)(1-\tilde{c}_1)^2\right) - y\left(2-b^2\right)^2(1-\tilde{c}_1)(1-b)$$
$$-\frac{32\left(1-b^2\right)^2}{(b^2-4)^2}(1-b)(2-b)(1-\tilde{c}_1)^2 + \frac{64\left(1-b^2\right)^2}{(b^2-4)^2}y(1-\tilde{c}_1)(1-b)$$
$$+by\left(2-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1) - by^2\left(2-b^2\right)^2 - 32by\left(1-b^2\right)^2\frac{(2-b)(1-\tilde{c}_1)}{(b^2-4)^2} + \frac{64y^2b\left(1-b^2\right)^2}{(b^2-4)^2} > 0$$

$$\left((2-b-b^2)(1-\tilde{c}_1)\right)^2 b - y\left(2-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1) b - 16b\frac{\left(1-b^2\right)^2}{(b^2-4)^2}((2-b)(1-\tilde{c}_1))^2 + 32b\frac{\left(1-b^2\right)^2}{(b^2-4)^2}(2-b)(1-\tilde{c}_1)y$$
$$+(1-b)\left(2-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)^2 - y\left(2-b^2\right)^2(1-\tilde{c}_1)(1-b)$$
$$-\frac{32\left(1-b^2\right)^2}{(b^2-4)^2}(1-b)(2-b)(1-\tilde{c}_1)^2 + \frac{64\left(1-b^2\right)^2}{(b^2-4)^2}y(1-\tilde{c}_1)(1-b) > 0$$

Continuing to simplify:

$$\begin{aligned}
&\left(2-b-b^2\right)\left(2b-b^2-b^3\right)(1-\tilde{c}_1)^2 - y\left(2-b^2\right)\left(2b-b^2-b^3\right)(1-\tilde{c}_1) \\
&-16\frac{\left(1-b^2\right)^2}{(b^2-4)^2}\left(2b-b^2\right)(2-b)(1-\tilde{c}_1)^2 + 32\frac{\left(1-b^2\right)^2}{(b^2-4)^2}\left(2b-b^2\right)(1-\tilde{c}_1)\,y \\
&+(1-b)\left(2-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)^2 - y\left(2-b^2\right)(1-\tilde{c}_1)(1-b)\left(2-b^2\right) \\
&-\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}(2-2b)(2-b)(1-\tilde{c}_1)^2 + \frac{32\left(1-b^2\right)^2}{(b^2-4)^2}y(1-\tilde{c}_1)(2-2b) > 0 \\
&\left(2-b-b^2\right)\left(2b-b^2-b^3\right)(1-\tilde{c}_1)^2 - y\left(2-b^2\right)\left(2b-b^2-b^3\right)(1-\tilde{c}_1) \\
&-16\frac{\left(1-b^2\right)^2}{(b^2-4)^2}\left(2b-b^2\right)(2-b)(1-\tilde{c}_1)^2 + 32\frac{\left(1-b^2\right)^2}{(b^2-4)^2}\left(2b-b^2\right)(1-\tilde{c}_1)\,y \\
&+\left(2-2b-b^2+b^3\right)\left(2-b-b^2\right)(1-\tilde{c}_1)^2 - y\left(2-b^2\right)(1-\tilde{c}_1)\left(2-2b-b^2+b^3\right) \\
&-\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}(2-2b)(2-b)(1-\tilde{c}_1)^2 + \frac{32\left(1-b^2\right)^2}{(b^2-4)^2}y(1-\tilde{c}_1)(2-2b) > 0 \\
&\left(2-b-b^2\right)\left(2b-b^2-b^3+2-2b-b^2+b^3\right)(1-\tilde{c}_1)^2 \\
&-y\left(2-b^2\right)\left(2b-b^2-b^3+2-2b-b^2+b^3\right)(1-\tilde{c}_1) \\
&-16\frac{\left(1-b^2\right)^2}{(b^2-4)^2}\left(2b-b^2+2-2b\right)(2-b)(1-\tilde{c}_1)^2 + 32\frac{\left(1-b^2\right)^2}{(b^2-4)^2}\left(2b-b^2+2-2b\right)(1-\tilde{c}_1)\,y > 0 \\
&\left(2-b-b^2\right)\left(-2b^2+2\right)(1-\tilde{c}_1)^2 - y\left(2-b^2\right)\left(-2b^2+2\right)(1-\tilde{c}_1) \\
&-16\frac{\left(1-b^2\right)^2}{(b^2-4)^2}\left(2-b^2\right)(2-b)(1-\tilde{c}_1)^2 + 32\frac{\left(1-b^2\right)^2}{(b^2-4)^2}\left(2-b^2\right)(1-\tilde{c}_1)\,y > 0 \\
&\left(2-b-b^2\right)\left(-2b^2+2\right)(1-\tilde{c}_1) - y\left(2-b^2\right)\left(-2b^2+2\right) - 16\frac{\left(1-b^2\right)^2}{(b^2-4)^2}\left(2-b^2\right)(2-b)(1-\tilde{c}_1) + 32\frac{\left(1-b^2\right)^2}{(b^2-4)^2}\left(2-b^2\right)y > 0 \\
&y\left(32\frac{\left(1-b^2\right)^2}{(b^2-4)^2}\left(2-b^2\right) - \left(2-b^2\right)\left(-2b^2+2\right)\right) + \left(2-b-b^2\right)\left(-2b^2+2\right)(1-\tilde{c}_1) - 16\frac{\left(1-b^2\right)^2}{(b^2-4)^2}\left(2-b^2\right)(2-b)(1-\tilde{c}_1) > 0 \\
&y\left(32\frac{\left(1-b^2\right)^2}{(b^2-4)^2}\left(2-b^2\right) - \left(2-b^2\right)\left(2-2b^2\right)\right) + (1-\tilde{c}_1)\left(\left(2-b-b^2\right)\left(2-2b^2\right) - 16\frac{\left(1-b^2\right)^2}{(b^2-4)^2}\left(2-b^2\right)(2-b)\right) > 0 \\
&y\left(32\frac{\left(1-b^2\right)^2}{(b^2-4)^2}\left(2-b^2\right) - 2\left(2-b^2\right)\left(1-b^2\right)\right) + (1-\tilde{c}_1)\left(2\left(2-b-b^2\right)\left(1-b^2\right) - 16\frac{\left(1-b^2\right)^2}{(b^2-4)^2}\left(2-b^2\right)(2-b)\right) > 0 \\
&y\left(32\frac{\left(1-b^2\right)}{(b^2-4)^2}\left(2-b^2\right) - 2\left(2-b^2\right)\right) + (1-\tilde{c}_1)\left(2\left(2-b-b^2\right) - 16\frac{\left(1-b^2\right)}{(b^2-4)^2}\left(2-b^2\right)(2-b)\right) > 0 \\
&y\left(16\frac{\left(1-b^2\right)}{(b^2-4)^2}\left(2-b^2\right) - \left(2-b^2\right)\right) + (1-\tilde{c}_1)\left(\left(2-b-b^2\right) - 8\frac{\left(1-b^2\right)}{(b^2-4)^2}\left(2-b^2\right)(2-b)\right) > 0.
\end{aligned}$$

Thus, $\frac{\partial}{\partial y}A_2(\tilde{c}_1, y, 0) > 0$ if

$$F_1(y) = \left(\underbrace{\left(2-b^2\right)\frac{16\left(1-b^2\right)}{\left(b^2-4\right)^2} - \left(2-b^2\right)}_{\text{Term 1}}\right) y + (1-\tilde{c}_1)\left(\underbrace{(1-b)(2+b) - (2-b)\frac{8\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-b^2\right)}_{\text{Term 2}}\right) > 0.$$

Note that Term 1 is negative by

$$\begin{aligned}
16\frac{\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-b^2\right) - \left(2-b^2\right) &< 0 \\
16\frac{\left(1-b^2\right)}{\left(b^2-4\right)^2} &< 1 \\
16\left(1-b^2\right) &< \left(b^2-4\right)^2 \\
16\left(1-b^2\right) &< \left(4-b^2\right)^2 \\
16-16b^2 &< 16-8b^2+b^4 \\
0 &< 8b^2+b^4
\end{aligned}$$

and Term 2 is positive by

$$
\begin{aligned}
(1-b)(2+b) &> (2-b)\frac{8\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-b^2\right) \\
\left(2-b-b^2\right) &> 8\frac{\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-b^2\right)(2-b) \\
\left(2-b-b^2\right)\left(b^2-4\right)^2 &> 8\left(2-b^2\right)(2-b)\left(1-b^2\right) \\
\left(2-b-b^2\right)(b-2)^2(b+2)^2 &> 8\left(2-b^2\right)(2-b)\left(1-b^2\right) \\
\left(2-b-b^2\right)(2-b)^2(b+2)^2 &> 8\left(2-b^2\right)(2-b)\left(1-b^2\right) \\
\left(2-b-b^2\right)(2-b)(2+b)^2 &> 8\left(2-b^2\right)\left(1-b^2\right) \\
b^5+3b^4-4b^3-16b^2+16 &> 8b^4-24b^2+16 \\
b^5+8b^2 &> 5b^4+4b^3 \\
b^3+8 &> 5b^2+4b \\
b^3+8-5b^2-4b &> 0 \\
(1-b)\left(8+4b-b^2\right) &> 0 \\
8+4b-b^2 &> 0
\end{aligned}
$$

Thus, $F_1(y)$ is decreasing in y. Therefore, it is sufficient to show that $F_1((1-b)(1-\tilde{c}_1)) > 0$ because $y_U < (1-b)(1-\tilde{c}_1)$ (where the inequality follows from Lemma B.1) and $F_1(y)$ is decreasing in y.

$$F_1((1-b)(1-\tilde{c}_1)) > 0$$

$$(1-\tilde{c}_1)(1-b)\left(16\frac{(1-b^2)}{(b^2-4)^2}(2-b^2)-(2-b^2)\right)+(1-\tilde{c}_1)\left((2-b-b^2)-8\frac{(1-b^2)}{(b^2-4)^2}(2-b^2)(2-b)\right) > 0$$

$$(1-b)\left(16\frac{(1-b^2)}{(b^2-4)^2}(2-b^2)-(2-b^2)\right)+\left((2-b-b^2)-8\frac{(1-b^2)}{(b^2-4)^2}(2-b^2)(2-b)\right) > 0$$

$$(1-b)\left(16\frac{(1-b^2)}{(b^2-4)^2}(2-b^2)-(2-b^2)\right)+\left((1-b)(2+b)-8\frac{(1-b)(1+b)}{(b^2-4)^2}(2-b^2)(2-b)\right) > 0$$

$$\left(16\frac{(1-b^2)}{(b^2-4)^2}(2-b^2)-(2-b^2)\right)+\left((2+b)-8\frac{(1+b)}{(b^2-4)^2}(2-b^2)(2-b)\right) > 0$$

$$\left(16\frac{(1-b^2)}{(b^2-4)^2}(2-b^2)-2+b^2\right)+\left(2+b-8\frac{(1+b)}{(b^2-4)^2}(2-b^2)(2-b)\right) > 0$$

$$16\frac{(1-b^2)}{(b^2-4)^2}(2-b^2)+b^2+b-8\frac{(1+b)}{(b^2-4)^2}(2-b^2)(2-b) > 0$$

$$8\frac{(2-b^2)}{(b^2-4)^2}\left(2(1-b^2)-(1+b)(2-b)\right)+b^2+b > 0$$

$$8\frac{(2-b^2)}{(b^2-4)^2}\left(2-2b^2-(2+b-b^2)\right)+b^2+b > 0$$

$$8\frac{(2-b^2)}{(b^2-4)^2}\left(-b^2-b\right)+b^2+b > 0$$

$$-8\frac{(2-b^2)}{(b^2-4)^2}\left(b^2+b\right)+b^2+b > 0$$

$$-8\frac{(2-b^2)}{(b^2-4)^2}+1 > 0$$

$$1 > 8\frac{(2-b^2)}{(b^2-4)^2}$$

$$(b^2-4)^2 > 8(2-b^2)$$

$$b^4-8b^2+16 > 16-9b^2$$

$$b^4+b^2 > 0$$

which holds as $b<1$. □

Lemma B.3. *$A_1(\tilde{c}_1,x,a)=\frac{b^2((1-\tilde{c}_1)(1-b)-(x-a))^2}{((2-b-b^2)(1-\tilde{c}_1)+b(x-a))^2-16(b^2-1)^2\left(\frac{((2-b)(1-\tilde{c}_1)+b(x-a))^2}{(b^2-4)^2}\right)}$ is decreasing in $x-a$.*

Proof. Note that $A_1(\tilde{c}_1,x,a)=A_1(\tilde{c}_1,y,0)$ where y is defined as $y=x-a$. Thus, the remainder of the proof, for this case, shows that $\frac{\partial}{\partial y}A_1(\tilde{c}_1, y, 0) < 0$ for y such that $y \in (y_L, y_U)$. $\frac{\partial}{\partial y}A_1(\tilde{c}_1, y, 0) < 0$ when

$$-\left(\left(\left(2-b-b^2\right)(1-\tilde{c}_1)+yb\right)^2-16\left(1-b^2\right)^2\frac{((2-b)(1-\tilde{c}_1)+by)^2}{(b^2-4)^2}\right)2b^2((1-\tilde{c}_1)(1-b)-y)$$
$$+b^2((1-\tilde{c}_1)(1-b)-y)^2\left(-2b\left(\left(2-b-b^2\right)(1-\tilde{c}_1)+by\right)+32b\left(1-b^2\right)^2\frac{((2-b)(1-\tilde{c}_1)+by)}{(b^2-4)^2}\right)<0$$

$$2\left(\left(\left(2-b-b^2\right)(1-\tilde{c}_1)+yb\right)^2-16\left(1-b^2\right)^2\frac{((2-b)(1-\tilde{c}_1)+by)^2}{(b^2-4)^2}\right)$$
$$+((1-\tilde{c}_1)(1-b)-y)\left(-2b\left(\left(2-b-b^2\right)(1-\tilde{c}_1)+by\right)+32b\left(1-b^2\right)^2\frac{((2-b)(1-\tilde{c}_1)+by)}{(b^2-4)^2}\right)>0$$

$$2\left(\left(\left(2-b-b^2\right)(1-\tilde{c}_1)\right)^2+2yb\left(2-b-b^2\right)(1-\tilde{c}_1)+y^2b^2-16\frac{\left(1-b^2\right)^2}{(b^2-4)^2}\left(((2-b)(1-\tilde{c}_1))^2+2b(2-b)(1-\tilde{c}_1)y+y^2b^2\right)\right)$$
$$-((1-\tilde{c}_1)(1-b)-y)\left(-2b\left(\left(2-b-b^2\right)(1-\tilde{c}_1)+by\right)+32b\left(1-b^2\right)^2\frac{((2-b)(1-\tilde{c}_1)+by)}{(b^2-4)^2}\right)>0$$

$$2\left(\left(2-b-b^2\right)(1-\tilde{c}_1)\right)^2+4yb\left(2-b-b^2\right)(1-\tilde{c}_1)+2y^2b^2$$
$$-32\frac{\left(1-b^2\right)^2}{(b^2-4)^2}((2-b)(1-\tilde{c}_1))^2-64b\frac{\left(1-b^2\right)^2}{(b^2-4)^2}(2-b)(1-\tilde{c}_1)y-32\frac{\left(1-b^2\right)^2}{(b^2-4)^2}y^2b^2$$
$$+2b\left(2-b-b^2\right)(1-\tilde{c}_1)^2(1-b)+2b^2y(1-\tilde{c}_1)(1-b)-32(1-\tilde{c}_1)(1-b)b\left(1-b^2\right)^2\frac{(2-b)(1-\tilde{c}_1)}{(b^2-4)^2}-\frac{32b\left(1-b^2\right)^2}{(b^2-4)^2}by(1-\tilde{c}_1)(1-b)$$
$$-2b\left(2-b-b^2\right)(1-\tilde{c}_1)y-2b^2y^2+32yb\left(1-b^2\right)^2\frac{(2-b)(1-\tilde{c}_1)}{(b^2-4)^2}+\frac{32b\left(1-b^2\right)^2}{(b^2-4)^2}by^2>0$$

$$2\left(\left(2-b-b^2\right)(1-\tilde{c}_1)\right)^2+2yb\left(2-b-b^2\right)(1-\tilde{c}_1)-32\frac{\left(1-b^2\right)^2}{(b^2-4)^2}((2-b)(1-\tilde{c}_1))^2-32b\frac{\left(1-b^2\right)^2}{(b^2-4)^2}(2-b)(1-\tilde{c}_1)y$$
$$+2b\left(2-b-b^2\right)(1-\tilde{c}_1)^2(1-b)+2b^2y(1-\tilde{c}_1)(1-b)-32(1-\tilde{c}_1)(1-b)b\left(1-b^2\right)^2\frac{(2-b)(1-\tilde{c}_1)}{(b^2-4)^2}-\frac{32b\left(1-b^2\right)^2}{(b^2-4)^2}by(1-\tilde{c}_1)(1-b)>0$$

$$-y\left(32b\frac{\left(1-b^2\right)^2}{(b^2-4)^2}(2-b)(1-\tilde{c}_1)+\frac{32b\left(1-b^2\right)^2}{(b^2-4)^2}b(1-\tilde{c}_1)(1-b)-2b^2(1-\tilde{c}_1)(1-b)-2b\left(2-b-b^2\right)(1-\tilde{c}_1)\right)$$
$$+2\left(\left(2-b-b^2\right)(1-\tilde{c}_1)\right)^2-32\frac{\left(1-b^2\right)^2}{(b^2-4)^2}((2-b)(1-\tilde{c}_1))^2$$
$$+2b\left(2-b-b^2\right)(1-\tilde{c}_1)^2(1-b)-32(1-\tilde{c}_1)(1-b)b\left(1-b^2\right)^2\frac{(2-b)(1-\tilde{c}_1)}{(b^2-4)^2}>0$$

$$-y\left(32b\frac{(1-b)^2(1+b)^2}{(b^2-4)^2}(2-b)+\frac{32b(1-b)^2(1+b)^2}{(b^2-4)^2}b(1-b)-2b^2(1-b)-2b(2+b)(1-b)\right)$$
$$+2(2+b)^2(1-b)^2(1-\tilde{c}_1)-32\frac{(1-b)^2(1+b)^2}{(b^2-4)^2}((2-b))^2(1-\tilde{c}_1)$$
$$+2b\left(2-b-b^2\right)(1-\tilde{c}_1)(1-b)-32(1-\tilde{c}_1)(1-b)b(1-b)^2(1+b)^2\frac{(2-b)}{(b^2-4)^2}>0$$

$$-y\left(32b\frac{(1-b)(1+b)^2}{(b^2-4)^2}(2-b)+\frac{32b(1-b)^2(1+b)^2}{(b^2-4)^2}b-2b^2-2b(2+b)\right)$$
$$+2(2+b)^2(1-b)(1-\tilde{c}_1)-32\frac{(1-b)(1+b)^2}{(b^2-4)^2}((2-b))^2(1-\tilde{c}_1)$$
$$+2b\left(2-b-b^2\right)(1-\tilde{c}_1)-32(1-\tilde{c}_1)b(1-b)^2(1+b)^2\frac{(2-b)}{(b^2-4)^2}>0$$

$$-y\left(32b\frac{(1-b)(1+b)^2}{(b^2-4)^2}((2-b)+b(1-b))-2b^2-4b-2b^2\right)$$
$$+2\left(2-b-b^2\right)(1-\tilde{c}_1)(2+2b)-32\frac{(1-b)(1+b)^2(2-b)}{(b^2-4)^2}(1-\tilde{c}_1)(2-b+b(1-b))>0$$

$$-y\left(32b\frac{(1-b)(1+b)^2}{(b^2-4)^2}\left(2-b^2\right)-4b(1+b)\right)$$
$$+4\left(2-b-b^2\right)(1-\tilde{c}_1)(1+b)-32\frac{(1-b)(1+b)^2(2-b)}{(b^2-4)^2}(1-\tilde{c}_1)\left(2-b^2\right)>0$$

$$-y\left(8b\frac{(1-b)(1+b)}{(b^2-4)^2}\left(2-b^2\right)-b\right)$$
$$+\left(2-b-b^2\right)(1-\tilde{c}_1)-8\frac{(1-b)(1+b)(2-b)}{(b^2-4)^2}(1-\tilde{c}_1)\left(2-b^2\right)>0$$

$$y\left(b-8b\frac{(1-b)(1+b)}{(b^2-4)^2}\left(2-b^2\right)\right)+(1-\tilde{c}_1)\left(\left(2-b-b^2\right)-8\frac{(1-b)(1+b)(2-b)}{(b^2-4)^2}\left(2-b^2\right)\right)>0$$

Thus, $\frac{\partial}{\partial y} A_1(\tilde{c}_1, y, 0) < 0$ if

$$F_2(y) = y\left(\underbrace{b - 8b\frac{(1-b)(1+b)}{(b^2-4)^2}\left(2-b^2\right)}_{\text{Term 1}}\right) + (1-\tilde{c}_1)\left(\underbrace{\left(2-b-b^2\right) - 8\frac{(1-b)(1+b)(2-b)}{(b^2-4)^2}\left(2-b^2\right)}_{\text{Term 2}}\right) > 0.$$

Note that Term 1 is positive by

$$\begin{aligned}
8b\frac{(1-b)(1+b)}{(b^2-4)^2}\left(2-b^2\right) &< b \\
8\left(1-b^2\right)\left(2-b^2\right) &< \left(b^2-4\right)^2 \\
8\left(2-3b^2+b^4\right) &< b^4-8b^2+16 \\
16-24b^2+8b^4 &< b^4-8b^2+16 \\
-24b^2+8b^4 &< b^4-8b^2 \\
7b^4 &< 16b^2 \\
7b^2 &< 16
\end{aligned}$$

and Term 2 is positive by

$$\begin{aligned}
\left(2-b-b^2\right) - 8\frac{(1-b)(1+b)(2-b)}{(b^2-4)^2}\left(2-b^2\right) &> 0 \\
(2+b)(1-b) - 8\frac{(1-b)(1+b)(2-b)}{(b^2-4)^2}\left(2-b^2\right) &> 0 \\
(2+b) - 8\frac{(1+b)(2-b)}{(b^2-4)^2}\left(2-b^2\right) &> 0 \\
(2+b) - 8\frac{(1+b)(2-b)\left(2-b^2\right)}{(b^2-4)^2} &> 0 \\
(2+b)\left(b^2-4\right)^2 &> 8(1+b)(2-b)\left(2-b^2\right) \\
(2+b)(2-b)^2(2+b)^2 &> 8(1+b)(2-b)\left(2-b^2\right) \\
(2+b)(2-b)(2+b)^2 &> 8(1+b)\left(2-b^2\right) \\
(2+b)(2-b)\left(4+4b+b^2\right) &> 8\left(2-b^2+2b-b^3\right) \\
(2+b)\left(8+8b+2b^2-4b-4b^2-b^3\right) &> 16-8b^2+16b-8b^3 \\
(2+b)\left(8+8b+2b^2-4b-4b^2-b^3\right) &> 16-8b^2+16b-8b^3 \\
16+16b+4b^2-8b-8b^2-2b^3 & \\
+8b+8b^2+2b^3-4b^2-4b^3-b^4 &> 16-8b^2+16b-8b^3 \\
16+16b-4b^3-b^4 &> 16-8b^2+16b-8b^3 \\
8b^2+4b^3 &> b^4 \\
8+4b &> b^2.
\end{aligned}$$

Thus, $F_2(y)$ is increasing in y. Therefore, it is sufficient to show that $F_2(-(1-\tilde{c}_1)(1-b)) > 0$ because

$-(1-\tilde{c}_1)(1-b) < y_L$ and $F_2(y)$ is increasing in y.

$$
\begin{aligned}
F_2(-(1-\tilde{c}_1)(1-b)) &> 0 \\
-(1-\tilde{c}_1)(1-b)\left(b - 8b\frac{(1-b)(1+b)}{(b^2-4)^2}\left(2-b^2\right)\right) & \\
+(1-\tilde{c}_1)\left(\left(2-b-b^2\right) - 8\frac{(1-b)(1+b)(2-b)}{(b^2-4)^2}\left(2-b^2\right)\right) &> 0 \\
(1-b)\left(8b\frac{(1-b)(1+b)}{(b^2-4)^2}\left(2-b^2\right) - b\right) & \\
+\left((2+b)(1-b) - 8\frac{(1-b)(1+b)(2-b)}{(b^2-4)^2}\left(2-b^2\right)\right) &> 0 \\
\left(8b\frac{(1-b)(1+b)}{(b^2-4)^2}\left(2-b^2\right) - b\right) + \left((2+b) - 8\frac{(1+b)(2-b)}{(b^2-4)^2}\left(2-b^2\right)\right) &> 0 \\
8b\frac{(1-b)(1+b)}{(b^2-4)^2}\left(2-b^2\right) + 2 - 8\frac{(1+b)(2-b)}{(b^2-4)^2}\left(2-b^2\right) &> 0 \\
2 + 8\frac{(1-b)\left(2-b^2\right)}{(b^2-4)^2}\left(b^2+b-(2-b)\right) &> 0 \\
2 + 8\frac{(1-b)\left(2-b^2\right)}{(b^2-4)^2}\left(-2+2b+b^2\right) &> 0 \\
2 - 8\frac{(1-b)\left(2-b^2\right)}{(b^2-4)^2}(2+b)(1-b) &> 0
\end{aligned}
$$

$$
\begin{aligned}
2\left(b^2-4\right)^2 &> 8(1-b)\left(2-b^2\right)(2+b)(1-b) \\
\left(b^2-4\right)^2 &> 4(1-b)\left(2-b^2\right)(2+b)(1-b) \\
b^4 - 8b^2 + 16 &> -4b^5 + 20b^3 - 8b^2 - 24b + 16 \\
24b + b^4 + 4b^5 &> 20b^3 \\
24 + b^3 + 4b^4 &> 20b^2
\end{aligned}
$$

which holds for all b. □

B.2 Payoffs

Next, we derive manager payoffs in each phase.

B.2.1 Competitive Phase

The analysis of this subsection is based on Zanchettin (2006). In the competitive phase, manager 1 solves

$$
\begin{aligned}
&\max_{q_1} M_1(q_1, q_2) \\
&\max_{q_1} \left(P_1(q_1, q_2) - \tilde{c}_1\right) q_1 \\
&\max_{q_1} \left(1 - q_1 - bq_2 - \tilde{c}_1\right) q_1
\end{aligned}
$$

which yields a best reply function of $q_1(q_2) = \frac{1-\tilde{c}_1-bq_2}{2}$. Manager 2 solves

$$\max_{q_2} M_1(q_1,q_2)$$
$$\max_{q_2} \left(P_2(q_1,q_2) - \tilde{c}_2\right) q_2$$
$$\max_{q_2} \left(1 + a - q_2 - bq_1 - \tilde{c}_2\right) q_2$$

which yields a best reply function of $q_2(q_1) = \frac{1-\tilde{c}_2+a-bq_1}{2}$. Solving for the intersection of the best replies yields equilibrium quantities:

$$q_1^N = \frac{2-b-2\tilde{c}_1-ab+b\tilde{c}_2}{4-b^2} \tag{22}$$

and

$$q_2^N = \frac{2-b-2\tilde{c}_2+2a+b\tilde{c}_1}{4-b^2}. \tag{23}$$

Note that

$$\begin{aligned}
q_1^N &= \frac{2-b-2\tilde{c}_1-ab+b\tilde{c}_2}{4-b^2} > 0 \\
&\iff 2-b-2\tilde{c}_1-ab+b\tilde{c}_2 > 0 \\
&\iff 2-b-2\tilde{c}_1-ab+b\tilde{c}_2-b\tilde{c}_1+b\tilde{c}_1 > 0 \\
&\iff 2-b-2\tilde{c}_1+b\left(\tilde{c}_2-\tilde{c}_1-a\right)+b\tilde{c}_1 > 0 \\
&\iff (2-b)\left(1-\tilde{c}_1\right)+b\left(\tilde{c}_2-\tilde{c}_1-a\right) > 0 \\
&\iff \left(\frac{2}{b}-1\right)\left(1-\tilde{c}_1\right)+\left(\tilde{c}_2-\tilde{c}_1-a\right) > 0 \\
&\iff \tilde{c}_2-\tilde{c}_1-a > \left(1-\frac{2}{b}\right)\left(1-\tilde{c}_1\right)
\end{aligned}$$

and

$$\begin{aligned}
q_2^N &= \frac{2-b-2\tilde{c}_2+2a+b\tilde{c}_1}{4-b^2} > 0 \\
&\iff 2-b-2\tilde{c}_2+2a+b\tilde{c}_1 > 0 \\
&\iff 2-b-2\tilde{c}_2+2a+b\tilde{c}_1-2\tilde{c}_1+2\tilde{c}_1 > 0 \\
&\iff 2-b-2\left(\tilde{c}_2-\tilde{c}_1-a\right)+b\tilde{c}_1-2\tilde{c}_1 > 0 \\
&\iff (2-b)\left(1-\tilde{c}_1\right)-2\left(\tilde{c}_2-\tilde{c}_1-a\right) > 0 \\
&\iff \left(1-\frac{b}{2}\right)\left(1-\tilde{c}_1\right) > \tilde{c}_2-\tilde{c}_1-a.
\end{aligned}$$

$\tilde{c}_2-\tilde{c}_1-a > \left(1-\frac{2}{b}\right)\left(1-\tilde{c}_1\right)$ and $\left(1-\frac{b}{2}\right)\left(1-\tilde{c}_1\right) > \tilde{c}_2-\tilde{c}_1-a$ both hold as Assumption 2 implies

$$y_L < \tilde{c}_2-\tilde{c}_1-a < y_U$$
$$\implies \left(1-\tilde{c}_1\right)\left(1-\frac{2-b^2}{b}\right) < \tilde{c}_2-\tilde{c}_1-a < \left(1-\tilde{c}_1\right)\left(1-\frac{b}{2-b^2}\right)$$

by Lemma B.1. Additionally,

$$(1-\tilde{c}_1)\left(1-\frac{2-b^2}{b}\right) < \tilde{c}_2-\tilde{c}_1-a < (1-\tilde{c}_1)\left(1-\frac{b}{2-b^2}\right)$$
$$\implies \left(1-\frac{2}{b}\right)(1-\tilde{c}_1) < \tilde{c}_2-\tilde{c}_1-a < \left(1-\frac{b}{2}\right)(1-\tilde{c}_1).$$

Substituting (22) and (23) into the manager payoff functions yields:

$$M_1^N = \frac{\left(2\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)\right)^2}{\left(4-b^2\right)^2}$$

and

$$M_2^N = \frac{\left(2\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)\right)^2}{\left(4-b^2\right)^2}.$$

B.2.2 Collusive Phase

Managers collude by setting quantities to maximize their joint pay:

$$\max_{q_1,q_2} M_1(q_1,q_2)+M_2(q_1,q_2)$$

which yields collusive quantities of

$$q_1^C = \frac{1-\tilde{c}_1-b\left(1-\tilde{c}_2+a\right)}{2\left(1-b^2\right)}$$

and

$$q_2^C = \frac{1-\tilde{c}_2+a-b\left(1-\tilde{c}_1\right)}{2\left(1-b^2\right)}.$$

Collusive payoffs are

$$M_1^C = \frac{\left(1-\tilde{c}_1\right)\left(1-\tilde{c}_1-b\left(1-\tilde{c}_2+a\right)\right)}{4\left(1-b^2\right)}$$

and

$$M_2^C = \frac{\left(1-\tilde{c}_2+a\right)\left(1-\tilde{c}_2+a-b\left(1-\tilde{c}_1\right)\right)}{4\left(1-b^2\right)}.$$

B.2.3 Defection Phase

Manager 1 Manager 1 solves, when defecting,

$$\begin{aligned}\max_q M_1(q,q_2^C) &= \max_q q(P_1(q,q_2^C)-\tilde{c}_1)\\ &= \max_q q\left(\left[1-q-bq_2^C\right]-\tilde{c}_1\right)\end{aligned}$$

which yields a defection quantity of $q_1^D = \frac{1-bq_2^C-\tilde{c}_1}{2}$ and defection profits of

$$M_1(q_1^D,q_2^C) = \frac{\left(\left(2-b^2\right)\left(1-\tilde{c}_1\right)-b\left(1-\tilde{c}_2+a\right)\right)^2}{16\,\left(1-b^2\right)^2}.$$

Manager 2 Manager 2 solves, when defecting,

$$\begin{aligned}\max_q M_2(q_1^C,q_2) &= \max_q q(P_2(q_1^C,q)-\tilde{c}_2)\\ &= \max_q q(\left(1+a-q-bq_1^C\right)-\tilde{c}_2)\end{aligned}$$

which yields a defection quantity of $q_2^D = \frac{1+a-bq_1^C-\tilde{c}_2}{2}$ and defection profits of

$$M_2(q_1^C,q_2^D) = \frac{\left(\left(2-b^2\right)\left(1-\tilde{c}_2+a\right)-b\left(1-\tilde{c}_1\right)\right)^2}{16\,\left(1-b^2\right)^2}.$$

B.3 Critical Discount Factor

Manager 1:

The critical discount factor for manager 1 is

$$\begin{aligned}
\delta_1^* &= \frac{M_1(q_1^D, q_2^C) - M_1(q_1^C, q_2^C)}{M_1(q_1^D, q_2^C) - M_1(q_1^N, q_2^N)} \\
&= \frac{\frac{\left(\left(2-b^2\right)(1-\tilde{c}_1)-b(1-\tilde{c}_2+a)\right)^2}{16\left(1-b^2\right)^2} - \frac{(1-\tilde{c}_1)(1-\tilde{c}_1-b(1-\tilde{c}_2+a))}{4\left(1-b^2\right)}}{\frac{\left(\left(2-b^2\right)(1-\tilde{c}_1)-b(1-\tilde{c}_2+a)\right)^2}{16\left(1-b^2\right)^2} - \frac{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{\left(4-b^2\right)^2}} \\
&= \frac{\left(\left(2-b^2\right)(1-\tilde{c}_1) - b(1-\tilde{c}_2+a)\right)^2 - 4\left(1-b^2\right)(1-\tilde{c}_1)(1-\tilde{c}_1 - b(1-\tilde{c}_2+a))}{((2-b^2)(1-\tilde{c}_1) - b(1-\tilde{c}_2+a))^2 - 16(1-b^2)^2 \frac{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{\left(4-b^2\right)^2}} \\
&= \frac{\left(2-b^2\right)^2(1-\tilde{c}_1)^2 + b^2(1-\tilde{c}_2+a)^2 - 2\left(2-b^2\right)b(1-\tilde{c}_1)(1-\tilde{c}_2+a) - 4\left(1-b^2\right)(1-\tilde{c}_1)(1-\tilde{c}_1-b(1-\tilde{c}_2+a))}{((2-b^2)(1-\tilde{c}_1) - b(1-\tilde{c}_2+a))^2 - 16(1-b^2)^2 \frac{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{\left(4-b^2\right)^2}} \\
&= \frac{\left(4-4b^2+b^4\right)(1-\tilde{c}_1)^2 + b^2(1-\tilde{c}_2+a)^2 - \left(4b-2b^3\right)(1-\tilde{c}_1)(1-\tilde{c}_2+a) - 4\left(1-b^2\right)(1-\tilde{c}_1)(1-\tilde{c}_1-b(1-\tilde{c}_2+a))}{((2-b^2)(1-\tilde{c}_1) - b(1-\tilde{c}_2+a))^2 - 16(1-b^2)^2 \frac{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{\left(4-b^2\right)^2}} \\
&= \frac{b^4(1-\tilde{c}_1)^2 + b^2(1-\tilde{c}_2+a)^2 - 2b^3(1-\tilde{c}_1)(1-\tilde{c}_2+a)}{((2-b^2)(1-\tilde{c}_1) - b(1-\tilde{c}_2+a))^2 - 16(1-b^2)^2 \frac{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{\left(4-b^2\right)^2}} \\
&= b^2 \frac{b^2(1-\tilde{c}_1)^2 + (1-\tilde{c}_2+a)^2 - 2b(1-\tilde{c}_1)(1-\tilde{c}_2+a)}{((2-b^2)(1-\tilde{c}_1) - b(1-\tilde{c}_2+a))^2 - 16(1-b^2)^2 \frac{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{\left(4-b^2\right)^2}} \\
&= b^2 \frac{((1-\tilde{c}_2+a) - b(1-\tilde{c}_1))^2}{((2-b^2)(1-\tilde{c}_1) - b(1-\tilde{c}_2+a))^2 - 16(1-b^2)^2 \frac{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{\left(4-b^2\right)^2}}.
\end{aligned}$$

Manager 2:

The critical discount factor for manager 2 is

$$\begin{aligned}
\delta_2^* &= \frac{M_2(q_1^C, q_2^D) - M_2(q_1^C, q_2^C)}{M_2(q_1^C, q_2^D) - M_2(q_1^N, q_2^N)} \\
&= \frac{\frac{\left(\left(2-b^2\right)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1)\right)^2}{16\left(1-b^2\right)^2} - \frac{(1-\tilde{c}_2+a)(1-\tilde{c}_2+a-b(1-\tilde{c}_1))}{4\left(1-b^2\right)}}{\frac{\left(\left(2-b^2\right)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1)\right)^2}{16\left(1-b^2\right)^2} - \frac{(2(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{\left(4-b^2\right)^2}} \\
&= \frac{\left(\left(2-b^2\right)(1-\tilde{c}_2+a) - b(1-\tilde{c}_1)\right)^2 - 4\left(1-b^2\right)(1-\tilde{c}_2+a)(1-\tilde{c}_2+a-b(1-\tilde{c}_1))}{((2-b^2)(1-\tilde{c}_2+a) - b(1-\tilde{c}_1))^2 - 16(1-b^2)^2 \frac{(2(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{\left(4-b^2\right)^2}} \\
&= \frac{\left(2-b^2\right)^2(1-\tilde{c}_2+a) - 2b\left(2-b^2\right)(1-\tilde{c}_1)(1-\tilde{c}_2+a) + b^2(1-\tilde{c}_1)^2 - 4\left(1-b^2\right)(1-\tilde{c}_2+a)(1-\tilde{c}_2+a-b(1-\tilde{c}_1))}{((2-b^2)(1-\tilde{c}_2+a) - b(1-\tilde{c}_1))^2 - 16(1-b^2)^2 \frac{(2(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{\left(4-b^2\right)^2}} \\
&= \frac{\left(4-4b^2+b^4\right)(1-\tilde{c}_2+a) - \left(4b-2b^3\right)(1-\tilde{c}_1)(1-\tilde{c}_2+a) + b^2(1-\tilde{c}_1)^2 - 4\left(1-b^2\right)(1-\tilde{c}_2+a)(1-\tilde{c}_2+a-b(1-\tilde{c}_1))}{((2-b^2)(1-\tilde{c}_2+a) - b(1-\tilde{c}_1))^2 - 16(1-b^2)^2 \frac{(2(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{\left(4-b^2\right)^2}} \\
&= \frac{b^4(1-\tilde{c}_2+a) - 2b^3(1-\tilde{c}_1)(1-\tilde{c}_2+a) + b^2(1-\tilde{c}_1)^2}{((2-b^2)(1-\tilde{c}_2+a) - b(1-\tilde{c}_1))^2 - 16(1-b^2)^2 \frac{(2(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{\left(4-b^2\right)^2}} \\
&= b^2 \frac{((1-\tilde{c}_1) - b(1-\tilde{c}_2+a))^2}{((2-b^2)(1-\tilde{c}_2+a) - b(1-\tilde{c}_1))^2 - 16(1-b^2)^2 \frac{(2(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{\left(4-b^2\right)^2}}.
\end{aligned}$$

Lemma B.4. *i)* $\delta_1^*(\theta_1, \theta_2) = \delta_2^*(\theta_1, \theta_2)$ *when* $x - a = 0$*, ii)* $\delta_2^*(\theta_1, \theta_2) > \delta_1^*(\theta_1, \theta_2)$ *when* $x - a > 0$ *and iii)* $\delta_2^*(\theta_1, \theta_2) < \delta_1^*(\theta_1, \theta_2)$ *when* $x - a < 0$*.*

Proof. Part i) Note that $\delta_1^*(\theta_1, \theta_2) = \delta_2^*(\theta_1, \theta_2) = \frac{(2+b)^2}{b^2+8b+8}$ when $x - a = 0$ because firms are effectively symmetric.

Part ii) Consider the case of $x - a > 0$. By Lemma B.3, $\delta_1^*(\theta_1, \theta_2)$ is decreasing in $x - a$ and, thus, $\delta_1^*(\theta_1, \theta_2) < \frac{(2+b)^2}{b^2+8b+8}$. By Lemma B.2, $\delta_2^*(\theta_1, \theta_2)$ is increasing in $x - a$ and, thus, $\delta_2^*(\theta_1, \theta_2) > \frac{(2+b)^2}{b^2+8b+8}$. Thus, $\delta_2^*(\theta_1, \theta_2) > \frac{(2+b)^2}{b^2+8b+8} > \delta_1^*(\theta_1, \theta_2)$.

Part iii) Consider the case of $x - a < 0$. By Lemma B.3, $\delta_1^*(\theta_1, \theta_2)$ is decreasing in $x - a$ and, thus, $\delta_1^*(\theta_1, \theta_2) > \frac{(2+b)^2}{b^2+8b+8}$. By Lemma B.2, $\delta_2^*(\theta_1, \theta_2)$ is increasing in $x - a$ and, thus, $\delta_2^*(\theta_1, \theta_2) < \frac{(2+b)^2}{b^2+8b+8}$. Thus, $\delta_1^*(\theta_1, \theta_2) > \frac{(2+b)^2}{b^2+8b+8} > \delta_2^*(\theta_1, \theta_2)$. □

By Lemma B.4, the industry critical discount factor equals

$$\delta^*(\theta_1, \theta_2) = \begin{cases} \frac{b^2((1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{((2-b^2)(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2-16(1-b^2)^2\frac{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{(4-b^2)^2}} & \text{if } x-a<0 \\ \frac{b^2((1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{((2-b^2)(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2-16(1-b^2)^2\frac{(2(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{(4-b^2)^2}} & \text{if } x-a\geq 0. \end{cases}$$

Note that when compensation structures are identical (i.e., $\theta_1 = \theta_2$) and firms are homogenous (i.e., $c_1 = c_2$ and $a = 0$), the critical discount factor is

$$\delta^*(\theta_1, \theta_2) = \frac{(2+b)^2}{b^2+8b+8}.$$

B.4 Assumptions

Throughout this section, we assume Assumption 2 holds. We show below that Assumption 2 implies $\delta^*(\theta_1, \theta_2) < 1$ holds.

Lemma B.5. $\delta^*(\theta_1, \theta_2) < 1$ *if and only if Assumption 2 holds.*

Proof. ($\Longleftarrow$)Assume Assumption 1 holds. Collusion is sustainable for some discount factor less than 1 if

$$\frac{M_i^C(\theta_1, \theta_2)}{1-\delta} \geq M_i^D(\theta_1, \theta_2) + \delta \frac{M_i^N(\theta_1, \theta_2)}{1-\delta}.$$

If $M_1^C(\theta_1, \theta_2) > M_1^N(\theta_1, \theta_2)$ and $M_2^C(\theta_1, \theta_2) > M_2^N(\theta_1, \theta_2)$, then the above inequality is satisfied for some δ sufficiently close to 1 for both managers. It remains to show Assumption 1 implies $M_1^C(\theta_1, \theta_2) > M_1^N(\theta_1, \theta_2)$ and $M_2^C(\theta_1, \theta_2) > M_2^N(\theta_1, \theta_2)$. Note that

$$\begin{aligned} & M_1^C(\theta_1, \theta_2) > M_1^N(\theta_1, \theta_2) \\ \Longleftrightarrow\quad & \frac{(1-\tilde{c}_1)(1-\tilde{c}_1-b(1-\tilde{c}_2+a))}{4(1-b^2)} > \frac{(2(1-\tilde{c}_1)-b(1-\tilde{c}_2+a))^2}{(4-b^2)^2} \\ \Longleftrightarrow\quad & x-a > (1-\tilde{c}_1)\left(1+\frac{(8b+b^2\sqrt{b^2+8}+b^3-4\sqrt{b^2+8})}{8(1-b^2)}\right) \\ \text{and}\quad & x-a < (1-\tilde{c}_1)\left(1+\frac{(8b-b^2\sqrt{b^2+8}+b^3+4\sqrt{b^2+8})}{8(1-b^2)}\right). \end{aligned}$$

The first inequality holds by Assumption 1. The second inequality always holds by $\tilde{c}_2 < 1 + a$ as $\tilde{c}_2 <$

$1 + a \implies x - a < 1 - \tilde{c}_1$.

$$
\begin{aligned}
& M_2^C(\theta_1,\theta_2) > M_2^N(\theta_1,\theta_2) \\
\iff \quad & \frac{(1-\tilde{c}_2+a)(1-\tilde{c}_2+a-b(1-\tilde{c}_1))}{4(1-b^2)} > \frac{(2(1-\tilde{c}_2+a)-b(1-\tilde{c}_1))^2}{(4-b^2)^2} \\
\iff \quad & (1-\tilde{c}_1)\left(1+\frac{\left(-8b-b^3+b^2\sqrt{b^2+8}-4\sqrt{b^2+8}\right)}{2(b^2+8)}\right) > x-a \\
\text{or} \quad & (1-\tilde{c}_1)\left(1+\frac{\left(-8b-b^3-b^2\sqrt{b^2+8}+4\sqrt{b^2+8}\right)}{2(b^2+8)}\right) < x-a
\end{aligned}
$$

The first inequality holds by Assumption 1. The second inequality never holds by $\tilde{c}_2 < 1 + a$.[12] Thus, Assumption 1 implies $M_1^C(\theta_1,\theta_2) > M_1^N(\theta_1,\theta_2)$ and $M_2^C(\theta_1,\theta_2) > M_2^N(\theta_1,\theta_2)$.

($\implies$) Assume $\delta^*(\theta_1,\theta_2) < 1$. $\delta^*(\theta_1,\theta_2) < 1$ implies that there exists a $\delta \in (\delta^*(\theta_1,\theta_2), 1)$ such that

$$\frac{M_i^C(\theta_1,\theta_2)}{1-\delta} \geq M_i^D(\theta_1,\theta_2) + \delta\frac{M_i^N(\theta_1,\theta_2)}{1-\delta}$$

for both manager 1 and manager 2. Note that $M_i^D(\theta_1,\theta_2) > M_i^C(\theta_1,\theta_2)$. Thus, the above inequality implies $M_1^C(\theta_1,\theta_2) > M_1^N(\theta_1,\theta_2)$ and $M_2^C(\theta_1,\theta_2) > M_2^N(\theta_1,\theta_2)$ hold. As shown in the first part of the proof,

$$M_1^C(\theta_1,\theta_2) > M_1^N(\theta_1,\theta_2) \implies x-a > (1-\tilde{c}_1)\left(1+\frac{\left(8b+b^2\sqrt{b^2+8}+b^3-4\sqrt{b^2+8}\right)}{8(1-b^2)}\right)$$

and

$$M_2^C(\theta_1,\theta_2) > M_2^N(\theta_1,\theta_2) \implies (1-\tilde{c}_1)\left(1+\frac{\left(-8b-b^3+b^2\sqrt{b^2+8}-4\sqrt{b^2+8}\right)}{2(b^2+8)}\right) > x-a.$$

Thus, Assumption 1 holds. □

B.5 Proofs

In this section, we prove analogous results to Proposition 1-5 (from the main text) for the case of differentiated product quantity competition. Let $\delta_i^*(\tilde{c}_1, x, a)$ denote the critical discount factor of manager i when the perceived marginal cost of manager 1 is $\tilde{c}_1$, the asymmetry in perceived marginal cost is $x \equiv \tilde{c}_2 - \tilde{c}_1$ and the asymmetry in product quality is a. The industry critical discount factor $\delta^*(\tilde{c}_1, x, a)$ is defined analogously.

B.5.1 Proof of Proposition 1 (Quantity Competition)

Proposition. *i)* $\delta_1^*(\theta_1,\theta_2) = \delta_2^*(\theta_1,\theta_2)$ *when* $\tilde{c}_2 - \tilde{c}_1 - a = 0$,
ii) $\delta_2^*(\theta_1,\theta_2) > \delta_1^*(\theta_1,\theta_2)$ *when* $\tilde{c}_2 - \tilde{c}_1 - a > 0$, *and*
iii) $\delta_2^*(\theta_1,\theta_2) < \delta_1^*(\theta_1,\theta_2)$ *when* $\tilde{c}_2 - \tilde{c}_1 - a < 0$.

Proof. See the proof of Lemma B.4. □

B.5.2 Proof of Lemma 1 (Quantity Competition)

Lemma B.6. *Suppose* $x - a \neq 0$. *Then,* $\delta^*(\tilde{c}_1, x, a)$ *is increasing in* $\tilde{c}_1$ *under differentiated product quantity competition.*

---

[12] $\tilde{c}_2 < 1 + a \implies x - a < 1 - \tilde{c}_1 < (1-\tilde{c}_1)\left(1+\frac{\left(-8b-b^3-b^2\sqrt{b^2+8}+4\sqrt{b^2+8}\right)}{2(b^2+8)}\right)$ as $\frac{-8b-b^3-b^2\sqrt{b^2+8}+4\sqrt{b^2+8}}{2(b^2+8)} > 0$ for $b \in (0,1)$.

Proof. There are two cases to consider:

Case 1 ($x - a > 0$): In this case, the critical discount factor is determined by manager 2.[13] By the derivations in Section B.3, the critical discount factor for manager 2 is

$$\delta_2(\tilde{c}_1, x, a) = \frac{b^2\left((1-\tilde{c}_1)(1-b)+bx\right)^2}{\left((2-b-b^2)(1-\tilde{c}_1)-x\left(2-b^2\right)\right)^2 - \frac{16(1-b^2)^2}{(b^2-4)^2}\left((2-b)(1-\tilde{c}_1)-2x\right)^2}.$$

Note that $\delta_2(\tilde{c}_1, x, a) = \delta_2(\tilde{c}_1, y, 0)$ where y is defined as $y = x - a$. The remainder of the proof, for this case, shows that $\frac{\partial}{\partial \tilde{c}_1}\delta_2(\tilde{c}_1, y, 0) > 0$ for y such that $y \in (0, y_U)$. $\frac{\partial}{\partial \tilde{c}_1}\delta_2(\tilde{c}_1, y, 0) > 0$ when

$$\begin{aligned}
-2b^2(1-b)\left((1-\tilde{c}_1)(1-b)+by\right)&\left(\left((2-b-b^2)(1-\tilde{c}_1)-y\left(2-b^2\right)\right)^2 - \frac{16\left(1-b^2\right)^2}{(b^2-4)^2}\left((2-b)(1-\tilde{c}_1)-2y\right)^2\right)\\
-b^2\left((1-\tilde{c}_1)(1-b)+by\right)^2&\left(-2\left(2-b-b^2\right)\left((2-b-b^2)(1-\tilde{c}_1)-y\left(2-b^2\right)\right)+(2-b)\frac{32\left(1-b^2\right)^2}{(b^2-4)^2}\left((2-b)(1-\tilde{c}_1)-2y\right)\right) > 0
\end{aligned}$$

$$\begin{aligned}
(1-b)&\left(\left((2-b-b^2)(1-\tilde{c}_1)-y\left(2-b^2\right)\right)^2 - \frac{16\left(1-b^2\right)^2}{(b^2-4)^2}\left((2-b)(1-\tilde{c}_1)-2y\right)^2\right)\\
+\left((1-\tilde{c}_1)(1-b)+by\right)&\left(-\left(2-b-b^2\right)\left((2-b-b^2)(1-\tilde{c}_1)-y\left(2-b^2\right)\right)+(2-b)\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}\left((2-b)(1-\tilde{c}_1)-2y\right)\right) < 0
\end{aligned}$$

$$\begin{aligned}
(1-b)&\left(\left((2-b-b^2)(1-\tilde{c}_1)-y\left(2-b^2\right)\right)^2 - \frac{16\left(1-b^2\right)^2}{(b^2-4)^2}\left((2-b)(1-\tilde{c}_1)-2y\right)^2\right)\\
+\left((1-\tilde{c}_1)(1-b)\right)&\left(-\left(2-b-b^2\right)\left((2-b-b^2)(1-\tilde{c}_1)-y\left(2-b^2\right)\right)+(2-b)\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}\left((2-b)(1-\tilde{c}_1)-2y\right)\right)\\
+by&\left(-\left(2-b-b^2\right)\left((2-b-b^2)(1-\tilde{c}_1)-y\left(2-b^2\right)\right)+(2-b)\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}\left((2-b)(1-\tilde{c}_1)-2y\right)\right) < 0
\end{aligned}$$

$$\begin{aligned}
(1-b)&\left(\left((2-b-b^2)(1-\tilde{c}_1)-y\left(2-b^2\right)\right)^2 - \frac{16\left(1-b^2\right)^2}{(b^2-4)^2}\left((2-b)(1-\tilde{c}_1)-2y\right)^2\right)\\
&+\left((1-\tilde{c}_1)(1-b)\right)\left(-\left(2-b-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)+\left(2-b-b^2\right)y\left(2-b^2\right)\right)\\
&+\left((1-\tilde{c}_1)(1-b)\right)\left((2-b)^2\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}(1-\tilde{c}_1)-(2-b)\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}2y\right)\\
+by&\left(-\left(2-b-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)+\left(2-b-b^2\right)y\left(2-b^2\right)+(2-b)^2\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}(1-\tilde{c}_1)-(2-b)\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}2y\right) < 0
\end{aligned}$$

$$\begin{aligned}
&(1-b)\left(\left((2-b-b^2)(1-\tilde{c}_1)\right)^2 - 2y\left(2-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)\right)\\
&(1-b)\left(y^2\left(2-b^2\right)^2 - \frac{16\left(1-b^2\right)^2}{(b^2-4)^2}\left(\left((2-b)(1-\tilde{c}_1)\right)^2-4(2-b)(1-\tilde{c}_1)y+4y^2\right)\right)\\
&+\left((1-\tilde{c}_1)(1-b)\right)\left(-\left(2-b-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)+\left(2-b-b^2\right)y\left(2-b^2\right)\right)\\
&+\left((1-\tilde{c}_1)(1-b)\right)\left((2-b)^2\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}(1-\tilde{c}_1)-(2-b)\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}2y\right)\\
+by&\left(-\left(2-b-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)+\left(2-b-b^2\right)y\left(2-b^2\right)+(2-b)^2\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}(1-\tilde{c}_1)-(2-b)\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}2y\right) < 0.
\end{aligned}$$

[13]The critical discount factor for manager 2 is greater than the critical discount factor for manager 1 when $x - a > 0$ by Lemma B.4.

Continuing to simplify the condition yields:

$$
\begin{aligned}
&(1-b)\left(\left(\left(2-b-b^2\right)(1-\tilde{c}_1)\right)^2-2y\left(2-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)+y^2\left(2-b^2\right)^2\right)\\
&-(1-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}\left(((2-b)(1-\tilde{c}_1))^2-4(2-b)(1-\tilde{c}_1)\,y+4y^2\right)\\
&+((1-\tilde{c}_1)(1-b))\left(-\left(2-b-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)+\left(2-b-b^2\right)y\left(2-b^2\right)+(2-b)^2\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}(1-\tilde{c}_1)-(2-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}2y\right)\\
&+by\left(-\left(2-b-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)+\left(2-b-b^2\right)y\left(2-b^2\right)+(2-b)^2\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}(1-\tilde{c}_1)-(2-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}2y\right)<0\\
&(1-b)\left(\left(2-b-b^2\right)(1-\tilde{c}_1)\right)^2-2(1-b)\,y\left(2-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)+(1-b)\,y^2\left(2-b^2\right)^2\\
&-(1-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}((2-b)(1-\tilde{c}_1))^2+(1-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}4(2-b)(1-\tilde{c}_1)\,y-(1-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}4y^2\\
&+((1-\tilde{c}_1)(1-b))\left(-\left(2-b-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)+\left(2-b-b^2\right)y\left(2-b^2\right)+(2-b)^2\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}(1-\tilde{c}_1)-(2-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}2y\right)\\
&+by\left(-\left(2-b-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)+\left(2-b-b^2\right)y\left(2-b^2\right)+(2-b)^2\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}(1-\tilde{c}_1)-(2-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}2y\right)<0\\
&(1-b)\left(\left(2-b-b^2\right)(1-\tilde{c}_1)\right)^2-2(1-b)\,y\left(2-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)+(1-b)\,y^2\left(2-b^2\right)^2\\
&-(1-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}((2-b)(1-\tilde{c}_1))^2+(1-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}4(2-b)(1-\tilde{c}_1)\,y-(1-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}4y^2\\
&-\left(2-b-b^2\right)^2(1-\tilde{c}_1)^2(1-b)+\left(2-b-b^2\right)y\left(2-b^2\right)(1-\tilde{c}_1)(1-b)\\
&+(2-b)^2\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}(1-\tilde{c}_1)^2(1-b)-(2-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}2y(1-\tilde{c}_1)(1-b)\\
&+by\left(-\left(2-b-b^2\right)^2(1-\tilde{c}_1)+\left(2-b-b^2\right)y\left(2-b^2\right)+(2-b)^2\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}(1-\tilde{c}_1)-(2-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}2y\right)<0\\
&-(1-b)\,y\left(2-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)+(1-b)\,y^2\left(2-b^2\right)^2\\
&+(1-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}2(2-b)(1-\tilde{c}_1)\,y-(1-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}4y^2\\
&+by\left(-\left(2-b-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)+\left(2-b-b^2\right)y\left(2-b^2\right)+(2-b)^2\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}(1-\tilde{c}_1)-(2-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}2y\right)<0\\
&-(1-b)\,y\left(2-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)+(1-b)\,y^2\left(2-b^2\right)^2\\
&+(1-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}2(2-b)(1-\tilde{c}_1)\,y-(1-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}4y^2\\
&-b\left(2-b-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)\,y+\left(2-b-b^2\right)by^2\left(2-b^2\right)+(2-b)^2\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}(1-\tilde{c}_1)\,by-(2-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}2by^2<0\\
&-(1-b)\,y\left(2-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)+(1-b)\,y^2\left(2-b^2\right)^2\\
&+(1-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}2(2-b)(1-\tilde{c}_1)\,y-(1-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}4y^2\\
&-b\left(2-b-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)\,y+\left(2-b-b^2\right)by^2\left(2-b^2\right)+(2-b)^2\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}(1-\tilde{c}_1)\,by-(2-b)\,\frac{16\left(1-b^2\right)^2}{(b^2-4)^2}2by^2<0.
\end{aligned}
$$

Continuing to simplify yields:

$$y^2\left((1-b)\left(2-b^2\right)^2-(1-b)\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}4+\left(2-b-b^2\right)b\left(2-b^2\right)-(2-b)\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}2b\right)$$
$$+y\left(-(1-b)\left(2-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)+(1-b)\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}2(2-b)(1-\tilde{c}_1)\right)$$
$$+y\left(-b\left(2-b-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)+(2-b)^2\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}(1-\tilde{c}_1)b\right)<0$$
$$y^2\left((1-b)\left(2-b^2\right)^2-(1-b)\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}4+\left(2-b-b^2\right)b\left(2-b^2\right)-(2-b)\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}2b\right)$$
$$+y(1-\tilde{c}_1)\left(-(1-b)\left(2-b^2\right)\left(2-b-b^2\right)+(1-b)\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}2(2-b)-b\left(2-b-b^2\right)\left(2-b-b^2\right)+(2-b)^2\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}b\right)<0$$
$$y^2\left((1-b)\left(2-b^2\right)^2-(1-b)\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}4+(2+b)(1-b)b\left(2-b^2\right)-(2-b)\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}2b\right)$$
$$+y(1-\tilde{c}_1)\left(-(1-b)\left(2-b^2\right)(2+b)(1-b)+(1-b)\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}2(2-b)-b(2+b)^2(1-b)^2+(2-b)^2\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}b\right)<0$$
$$y^2\left(\underbrace{\left(2-b^2\right)^2-\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}4+(2+b)b\left(2-b^2\right)-(2-b)\frac{16(1-b)(1+b)^2}{\left(b^2-4\right)^2}2b}_{\text{Term 1}}\right)$$
$$+y(1-\tilde{c}_1)\left(\underbrace{-(1-b)\left(2-b^2\right)(2+b)+\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}2(2-b)-b(2+b)^2(1-b)+(2-b)^2\frac{16(1-b)(1+b)^2}{\left(b^2-4\right)^2}b}_{\text{Term 2}}\right)<0.$$

Term 1 simplifies to

$$\left(2-b^2\right)^2-\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}4+(2+b)b\left(2-b^2\right)-(2-b)\frac{16(1-b)(1+b)^2}{\left(b^2-4\right)^2}2b=$$
$$\left(2-b^2\right)^2+(2+b)b\left(2-b^2\right)-\frac{32\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2\left(1-b^2\right)+b(2-b)(1+b)\right)=$$
$$\left(2-b^2\right)^2+(2+b)b\left(2-b^2\right)-(1+b)\frac{32\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2(1-b)+b(2-b)\right)=$$
$$\left(2-b^2\right)^2+(2+b)b\left(2-b^2\right)-(1+b)\frac{32\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-2b+2b-b^2\right)=$$
$$\left(2-b^2\right)^2+(2+b)b\left(2-b^2\right)-(1+b)\frac{32\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-b^2\right)=$$
$$\left(2-b^2\right)\left[\left(2-b^2\right)+(2+b)b-(1+b)\frac{32\left(1-b^2\right)}{\left(b^2-4\right)^2}\right]=$$
$$\left(2-b^2\right)\left[2-b^2+2b+b^2-(1+b)\frac{32\left(1-b^2\right)}{\left(b^2-4\right)^2}\right]$$
$$\left(2-b^2\right)\left[2(1+b)-(1+b)\frac{32\left(1-b^2\right)}{\left(b^2-4\right)^2}\right]$$
$$2(1+b)\left[\left(2-b^2\right)-\left(2-b^2\right)\frac{16\left(1-b^2\right)}{\left(b^2-4\right)^2}\right].$$

Term 2 simplifies to

$$
\begin{aligned}
-(1-b)\left(2-b^2\right)(2+b)+\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}2(2-b)-b(2+b)^2(1-b)+(2-b)^2\,\frac{16(1-b)(1+b)^2}{\left(b^2-4\right)^2}b &= \\
-(1-b)\left(2-b^2\right)(2+b)-b(2+b)^2(1-b)+\frac{16\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2\left(1-b^2\right)(2-b)+(2-b)^2(1+b)\,b\right) &= \\
-(1-b)\left(2-b^2\right)(2+b)-b(2+b)^2(1-b)+(1+b)(2-b)\,\frac{16\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2(1-b)+(2-b)\,b\right) &= \\
-(1-b)\left(2-b^2\right)(2+b)-b(2+b)^2(1-b)+(1+b)(2-b)\,\frac{16\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-2b+2b-b^2\right) &= \\
-(1-b)\left(2-b^2\right)(2+b)-b(2+b)^2(1-b)+(1+b)(2-b)\,\frac{16\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-b^2\right) &= \\
-(1-b)(2+b)\left(\left(2-b^2\right)+b(2+b)\right)+(1+b)(2-b)\,\frac{16\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-b^2\right) &= \\
-(1-b)(2+b)\left(2-b^2+2b+b^2\right)+(1+b)(2-b)\,\frac{16\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-b^2\right) &= \\
-2(1-b)(2+b)(1+b)+(1+b)(2-b)\,\frac{16\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-b^2\right) &= \\
-2(1-b)(2+b)(1+b)+(1+b)(2-b)\,\frac{16\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-b^2\right) &= \\
2(1+b)\left(-(1-b)(2+b)+(2-b)\,\frac{8\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-b^2\right)\right)&.
\end{aligned}
$$

Thus, $\frac{\partial}{\partial \tilde{c}_1}\delta_2(\tilde{c}_1, y, 0) > 0$ if

$$
\begin{aligned}
&\left(2(1+b)\left[\left(2-b^2\right)-\left(2-b^2\right)\frac{16\left(1-b^2\right)}{\left(b^2-4\right)^2}\right]\right)y^2 \\
&+(1-\tilde{c}_1)\left(2(1+b)\left(-(1-b)(2+b)+(2-b)\,\frac{8\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-b^2\right)\right)\right)y<0
\end{aligned}
$$

or

$$
\left(\left(2-b^2\right)\frac{16\left(1-b^2\right)}{\left(b^2-4\right)^2}-\left(2-b^2\right)\right)y^2+(1-\tilde{c}_1)\left((1-b)(2+b)-(2-b)\,\frac{8\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-b^2\right)\right)y>0.
$$

Dividing both sides by y (which is positive) yields

$$
F_1(y)=\left(\underbrace{\left(2-b^2\right)\frac{16\left(1-b^2\right)}{\left(b^2-4\right)^2}-\left(2-b^2\right)}_{\text{Term 3}}\right)y+\left(\underbrace{(1-b)(2+b)-(2-b)\,\frac{8\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-b^2\right)}_{\text{Term 4}}\right)>0.
$$

Note that Term 3 is negative by

$$\begin{aligned}
16\frac{\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-b^2\right)-\left(2-b^2\right)&<0\\
16\frac{\left(1-b^2\right)}{\left(b^2-4\right)^2}&<1\\
16\left(1-b^2\right)&<\left(b^2-4\right)^2\\
16\left(1-b^2\right)&<\left(4-b^2\right)^2\\
16-16b^2&<16-8b^2+b^4\\
0&<8b^2+b^4
\end{aligned}$$

and Term 4 is positive by

$$\begin{aligned}
(1-b)(2+b)&>(2-b)\frac{8\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-b^2\right)\\
\left(2-b-b^2\right)&>8\frac{\left(1-b^2\right)}{\left(b^2-4\right)^2}\left(2-b^2\right)(2-b)\\
\left(2-b-b^2\right)\left(b^2-4\right)^2&>8\left(2-b^2\right)(2-b)\left(1-b^2\right)\\
\left(2-b-b^2\right)(b-2)^2(b+2)^2&>8\left(2-b^2\right)(2-b)\left(1-b^2\right)\\
\left(2-b-b^2\right)(2-b)^2(b+2)^2&>8\left(2-b^2\right)(2-b)\left(1-b^2\right)\\
\left(2-b-b^2\right)(2-b)(2+b)^2&>8\left(2-b^2\right)\left(1-b^2\right)\\
b^5+3b^4-4b^3-16b^2+16&>8b^4-24b^2+16\\
b^5+8b^2&>5b^4+4b^3\\
b^3+8&>5b^2+4b\\
b^3+8-5b^2-4b&>0\\
(1-b)\left(8+4b-b^2\right)&>0\\
8+4b-b^2&>0.
\end{aligned}$$

Thus, $F_1(y)$ is decreasing in y. Therefore, it is sufficient to show that $F_1((1-b)(1-\tilde{c}_1))>0$ because $y_U<(1-b)(1-\tilde{c}_1)$ (where the second inequality follows from Lemma B.1) and $F_1(y)$ is decreasing in y.

$$
\begin{aligned}
F_1((1-b)(1-\tilde{c}_1)) &> 0 \\
(1-\tilde{c}_1)(1-b)\left(16\frac{(1-b^2)}{(b^2-4)^2}(2-b^2)-(2-b^2)\right)+(1-\tilde{c}_1)\left((2-b-b^2)-8\frac{(1-b^2)}{(b^2-4)^2}(2-b^2)(2-b)\right) &> 0 \\
(1-b)\left(16\frac{(1-b^2)}{(b^2-4)^2}(2-b^2)-(2-b^2)\right)+\left((2-b-b^2)-8\frac{(1-b^2)}{(b^2-4)^2}(2-b^2)(2-b)\right) &> 0 \\
(1-b)\left(16\frac{(1-b^2)}{(b^2-4)^2}(2-b^2)-(2-b^2)\right)+\left((1-b)(2+b)-8\frac{(1-b)(1+b)}{(b^2-4)^2}(2-b^2)(2-b)\right) &> 0 \\
\left(16\frac{(1-b^2)}{(b^2-4)^2}(2-b^2)-(2-b^2)\right)+\left((2+b)-8\frac{(1+b)}{(b^2-4)^2}(2-b^2)(2-b)\right) &> 0 \\
\left(16\frac{(1-b^2)}{(b^2-4)^2}(2-b^2)-2+b^2\right)+\left(2+b-8\frac{(1+b)}{(b^2-4)^2}(2-b^2)(2-b)\right) &> 0 \\
16\frac{(1-b^2)}{(b^2-4)^2}(2-b^2)+b^2+b-8\frac{(1+b)}{(b^2-4)^2}(2-b^2)(2-b) &> 0 \\
8\frac{(2-b^2)}{(b^2-4)^2}\left(2(1-b^2)-(1+b)(2-b)\right)+b^2+b &> 0 \\
8\frac{(2-b^2)}{(b^2-4)^2}\left(2-2b^2-(2+b-b^2)\right)+b^2+b &> 0 \\
8\frac{(2-b^2)}{(b^2-4)^2}\left(-b^2-b\right)+b^2+b &> 0 \\
-8\frac{(2-b^2)}{(b^2-4)^2}\left(b^2+b\right)+b^2+b &> 0 \\
-8\frac{(2-b^2)}{(b^2-4)^2}+1 &> 0 \\
1 &> 8\frac{(2-b^2)}{(b^2-4)^2} \\
(b^2-4)^2 &> 8(2-b^2) \\
b^4-8b^2+16 &> 16-9b^2 \\
b^4+b^2 &> 0
\end{aligned}
$$

which holds as $b<1$.

Case 2 ($x-a<0$): In this case, the critical discount factor is determined by manager 1.[14] By Section B.3, the critical discount factor of manager 1 in this case is

$$
\delta_1(\tilde{c}_1,x,a)=\frac{b^2\left((1-\tilde{c}_1)(1-b)-(x-a)\right)^2}{\left((2-b-b^2)(1-\tilde{c}_1)+b(x-a)\right)^2-16(b^2-1)^2\left(\frac{((2-b)(1-\tilde{c}_1)+b(x-a))^2}{(b^2-4)^2}\right)}.
$$

Note that $\delta_1(\tilde{c}_1,x,a)=\delta_1(\tilde{c}_1,y,0)$ where y is defined as $y=x-a$. Thus, the remainder of the proof, for

[14] The critical discount factor for manager 1 is greater than the critical discount factor for manager 2 when $x-a<0$ by Lemma B.4.

this case, shows that $\frac{\partial}{\partial \tilde{c}_1}\delta_1(\tilde{c}_1, y, 0) > 0$ for y such that $y \in (y_L, 0)$. $\frac{\partial}{\partial \tilde{c}_1}\delta_1(\tilde{c}_1, y, 0) > 0$ when

$$-2b^2(1-b)((1-\tilde{c}_1)(1-b)-y)\left(\left((2-b-b^2)(1-\tilde{c}_1)+by\right)^2-\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}((2-b)(1-\tilde{c}_1)+by)^2\right)$$

$$-b^2((1-\tilde{c}_1)(1-b)-y)^2\left(-2\left(2-b-b^2\right)\left(\left(2-b-b^2\right)(1-\tilde{c}_1)+yb\right)+(2-b)\frac{32\left(1-b^2\right)^2}{\left(b^2-4\right)^2}((2-b)(1-\tilde{c}_1)+by)\right)>0$$

$$2(1-b)\left(\left((2-b-b^2)(1-\tilde{c}_1)+yb\right)^2-\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}((2-b)(1-\tilde{c}_1)+by)^2\right)$$

$$+((1-\tilde{c}_1)(1-b)-y)\left(-2\left(2-b-b^2\right)\left(\left(2-b-b^2\right)(1-\tilde{c}_1)+yb\right)+(2-b)\frac{32\left(1-b^2\right)^2}{\left(b^2-4\right)^2}((2-b)(1-\tilde{c}_1)+by)\right)<0$$

$$(1-b)\left(\left((2-b-b^2)(1-\tilde{c}_1)+yb\right)^2-\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}((2-b)(1-\tilde{c}_1)+by)^2\right)$$

$$+((1-\tilde{c}_1)(1-b))\left(-\left(2-b-b^2\right)\left(\left(2-b-b^2\right)(1-\tilde{c}_1)+yb\right)+(2-b)\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}((2-b)(1-\tilde{c}_1)+by)\right)$$

$$-y\left(-\left(2-b-b^2\right)\left(\left(2-b-b^2\right)(1-\tilde{c}_1)+yb\right)+(2-b)\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}((2-b)(1-\tilde{c}_1)+by)\right)<0$$

$$(1-b)\left(\left((2-b-b^2)(1-\tilde{c}_1)+yb\right)^2-\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}((2-b)(1-\tilde{c}_1)+by)^2\right)$$

$$+((1-\tilde{c}_1)(1-b))\left(-\left(2-b-b^2\right)\left(\left(2-b-b^2\right)(1-\tilde{c}_1)+yb\right)+(2-b)\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}((2-b)(1-\tilde{c}_1)+by)\right)$$

$$-y\left(-(1-b)(2+b)((1-b)(2+b)(1-\tilde{c}_1)+yb)+(2-b)\frac{16(1-b)^2(1+b)^2}{\left(b^2-4\right)^2}((2-b)(1-\tilde{c}_1)+by)\right)<0$$

$$\left(\left((2-b-b^2)(1-\tilde{c}_1)+yb\right)^2-\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}((2-b)(1-\tilde{c}_1)+by)^2\right)$$

$$+((1-\tilde{c}_1))\left(-\left(2-b-b^2\right)\left(\left(2-b-b^2\right)(1-\tilde{c}_1)+yb\right)+(2-b)\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}((2-b)(1-\tilde{c}_1)+by)\right)$$

$$-y\left(-(2+b)((1-b)(2+b)(1-\tilde{c}_1)+yb)+(2-b)\frac{16(1-b)(1+b)^2}{\left(b^2-4\right)^2}((2-b)(1-\tilde{c}_1)+by)\right)<0.$$

Continuing to simplify yields

$$\begin{aligned}
&\left(\left(\left(2-b-b^2\right)(1-\tilde{c}_1)+yb\right)^2-\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}\left((2-b)(1-\tilde{c}_1)+by\right)^2\right)\\
&+((1-\tilde{c}_1))\left(-\left(2-b-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)-\left(2-b-b^2\right)yb+(2-b)^2\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}(1-\tilde{c}_1)+(2-b)\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}by\right)\\
&-y\left(-(2+b)(1-b)(2+b)(1-\tilde{c}_1)-(2+b)yb+(2-b)^2\frac{16(1-b)(1+b)^2}{\left(b^2-4\right)^2}(1-\tilde{c}_1)+(2-b)\frac{16(1-b)(1+b)^2}{\left(b^2-4\right)^2}by\right)<0
\end{aligned}$$

$$\begin{aligned}
&\left(\left(\left(2-b-b^2\right)(1-\tilde{c}_1)\right)^2+2yb\left(2-b-b^2\right)(1-\tilde{c}_1)+y^2b^2-\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}\left(((2-b)(1-\tilde{c}_1))^2+2b(2-b)(1-\tilde{c}_1)y+b^2y^2\right)\right)\\
&+((1-\tilde{c}_1))\left(-\left(2-b-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)-\left(2-b-b^2\right)yb+(2-b)^2\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}(1-\tilde{c}_1)+(2-b)\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}by\right)\\
&-y\left(-(2+b)(1-b)(2+b)(1-\tilde{c}_1)-(2+b)yb+(2-b)^2\frac{16(1-b)(1+b)^2}{\left(b^2-4\right)^2}(1-\tilde{c}_1)+(2-b)\frac{16(1-b)(1+b)^2}{\left(b^2-4\right)^2}by\right)<0
\end{aligned}$$

$$\begin{aligned}
&\left(\left(2-b-b^2\right)(1-\tilde{c}_1)\right)^2+2yb\left(2-b-b^2\right)(1-\tilde{c}_1)+y^2b^2-\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}\left(((2-b)(1-\tilde{c}_1))^2+2b(2-b)(1-\tilde{c}_1)y+b^2y^2\right)\\
&-\left(2-b-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)^2-\left(2-b-b^2\right)yb(1-\tilde{c}_1)+(2-b)^2\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}(1-\tilde{c}_1)^2+(2-b)\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}by(1-\tilde{c}_1)\\
&+(2+b)(1-b)(2+b)(1-\tilde{c}_1)y+(2+b)y^2b-(2-b)^2\frac{16(1-b)(1+b)^2}{\left(b^2-4\right)^2}(1-\tilde{c}_1)y-(2-b)\frac{16(1-b)(1+b)^2}{\left(b^2-4\right)^2}by^2<0
\end{aligned}$$

$$\begin{aligned}
&\left(\left(2-b-b^2\right)(1-\tilde{c}_1)\right)^2+2yb\left(2-b-b^2\right)(1-\tilde{c}_1)+y^2b^2\\
&-\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}((2-b)(1-\tilde{c}_1))^2-\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}2b(2-b)(1-\tilde{c}_1)y-\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}b^2y^2\\
&-\left(2-b-b^2\right)\left(2-b-b^2\right)(1-\tilde{c}_1)^2-\left(2-b-b^2\right)yb(1-\tilde{c}_1)+(2-b)^2\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}(1-\tilde{c}_1)^2+(2-b)\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}by(1-\tilde{c}_1)\\
&+(2+b)(1-b)(2+b)(1-\tilde{c}_1)y+(2+b)y^2b-(2-b)^2\frac{16(1-b)(1+b)^2}{\left(b^2-4\right)^2}(1-\tilde{c}_1)y-(2-b)\frac{16(1-b)(1+b)^2}{\left(b^2-4\right)^2}by^2<0
\end{aligned}$$

$$\begin{aligned}
&yb\left(2-b-b^2\right)(1-\tilde{c}_1)+y^2b^2-\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}b(2-b)(1-\tilde{c}_1)y-\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}b^2y^2\\
&+(2+b)(1-b)(2+b)(1-\tilde{c}_1)y+(2+b)y^2b-(2-b)^2\frac{16(1-b)(1+b)^2}{\left(b^2-4\right)^2}(1-\tilde{c}_1)y-(2-b)\frac{16(1-b)(1+b)^2}{\left(b^2-4\right)^2}by^2<0.
\end{aligned}$$

Continuing to simplify yields:

$$y^2\left(b^2+(2+b)\,b-\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}b^2-(2-b)\,\frac{16\,(1-b)\,(1+b)^2}{\left(b^2-4\right)^2}b\right)$$
$$-y\left(\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}b\,(2-b)\,(1-\tilde{c}_1)-b\left(2-b-b^2\right)(1-\tilde{c}_1)\right)$$
$$-y\left(-\,(2+b)\,(1-b)\,(2+b)\,(1-\tilde{c}_1)+(2-b)^2\,\frac{16\,(1-b)\,(1+b)^2}{\left(b^2-4\right)^2}\,(1-\tilde{c}_1)\right)<0$$
$$y^2\left(2b^2+2b-\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}b^2-(2-b)\,\frac{16\,(1-b)\,(1+b)^2}{\left(b^2-4\right)^2}b\right)$$
$$-y\,(1-\tilde{c}_1)\left(\frac{16\,(1-b)\,(1+b)^2}{\left(b^2-4\right)^2}\left(b-b^2+2-b\right)(2-b)-(2+b)\,(1-b)\,(2+2b)\right)<0$$
$$y^2\left(2b(b+1)-\frac{16\left(1-b^2\right)^2}{\left(b^2-4\right)^2}b^2-(2-b)\,\frac{16\,(1-b)\,(1+b)^2}{\left(b^2-4\right)^2}b\right)$$
$$-y\,(1-\tilde{c}_1)\left(\frac{16\,(1-b)\,(1+b)^2}{\left(b^2-4\right)^2}\left(2-b^2\right)(2-b)-2\,(2+b)\,(1-b)\,(b+1)\right)<0$$
$$y^2\left(2b-\frac{16\,(1-b)^2\,(1+b)}{\left(b^2-4\right)^2}b^2-(2-b)\,\frac{16\,(1-b)\,(1+b)}{\left(b^2-4\right)^2}b\right)$$
$$-y\,(1-\tilde{c}_1)\left(\frac{16\,(1-b)\,(1+b)}{\left(b^2-4\right)^2}\left(2-b^2\right)(2-b)-2\,(2+b)\,(1-b)\right)<0$$
$$y^2\left(2b-\frac{16\,(1-b)\,(1+b)}{\left(b^2-4\right)^2}b\,(b(1-b)+2-b)\right)$$
$$-y\,(1-\tilde{c}_1)\,(1-b)\left(\frac{16\,(1+b)}{\left(b^2-4\right)^2}\left(2-b^2\right)(2-b)-2\,(2+b)\right)<0$$
$$y^2\left(2b-\frac{16\,(1-b)\,(1+b)}{\left(b^2-4\right)^2}b\left(2-b^2\right)\right)$$
$$-y\,(1-\tilde{c}_1)\,(1-b)\left(\frac{16\,(1+b)}{\left(b^2-4\right)^2}\left(2-b^2\right)(2-b)-2\,(2+b)\right)<0$$
$$y^2\left(b-\frac{8\,(1-b)\,(1+b)}{\left(b^2-4\right)^2}b\left(2-b^2\right)\right)+y\,(1-\tilde{c}_1)\,(1-b)\left((2+b)-\frac{8\,(1+b)}{\left(b^2-4\right)^2}\left(2-b^2\right)(2-b)\right)<0$$
$$y\left(b-\frac{8\,(1-b)\,(1+b)}{\left(b^2-4\right)^2}b\left(2-b^2\right)\right)+(1-\tilde{c}_1)\,(1-b)\left((2+b)-\frac{8\,(1+b)}{\left(b^2-4\right)^2}\left(2-b^2\right)(2-b)\right)>0.$$

Thus, the condition is

$$F_2(y)=y\left(\underbrace{b-8b\frac{(1-b)\,(1+b)}{\left(b^2-4\right)^2}\left(2-b^2\right)}_{\text{Term 1}}\right)+(1-\tilde{c}_1)\left(\underbrace{\left(2-b-b^2\right)-8\frac{(1-b)\,(1+b)\,(2-b)}{\left(b^2-4\right)^2}\left(2-b^2\right)}_{\text{Term 2}}\right)>0$$

Note that Term 1 is positive by

$$\begin{aligned}
8b\frac{(1-b)(1+b)}{\left(b^2-4\right)^2}\left(2-b^2\right) &< b \\
8\left(1-b^2\right)\left(2-b^2\right) &< \left(b^2-4\right)^2 \\
8\left(2-3b^2+b^4\right) &< b^4-8b^2+16 \\
16-24b^2+8b^4 &< b^4-8b^2+16 \\
-24b^2+8b^4 &< b^4-8b^2 \\
7b^4 &< 16b^2 \\
7b^2 &< 16
\end{aligned}$$

and Term 2 is positive by

$$\begin{aligned}
\left(2-b-b^2\right)-8\frac{(1-b)(1+b)(2-b)}{\left(b^2-4\right)^2}\left(2-b^2\right) &> 0 \\
(2+b)(1-b)-8\frac{(1-b)(1+b)(2-b)}{\left(b^2-4\right)^2}\left(2-b^2\right) &> 0 \\
(2+b)-8\frac{(1+b)(2-b)}{\left(b^2-4\right)^2}\left(2-b^2\right) &> 0 \\
(2+b)-8\frac{(1+b)(2-b)\left(2-b^2\right)}{\left(b^2-4\right)^2} &> 0 \\
(2+b)\left(b^2-4\right)^2 &> 8(1+b)(2-b)\left(2-b^2\right) \\
(2+b)(2-b)^2(2+b)^2 &> 8(1+b)(2-b)\left(2-b^2\right) \\
(2+b)(2-b)(2+b)^2 &> 8(1+b)\left(2-b^2\right) \\
(2+b)(2-b)\left(4+4b+b^2\right) &> 8\left(2-b^2+2b-b^3\right) \\
(2+b)\left(8+8b+2b^2-4b-4b^2-b^3\right) &> 16-8b^2+16b-8b^3 \\
(2+b)\left(8+8b+2b^2-4b-4b^2-b^3\right) &> 16-8b^2+16b-8b^3 \\
16+16b+4b^2-8b-8b^2-2b^3 & \\
+8b+8b^2+2b^3-4b^2-4b^3-b^4 &> 16-8b^2+16b-8b^3 \\
16+16b-4b^3-b^4 &> 16-8b^2+16b-8b^3 \\
8b^2+4b^3 &> b^4 \\
8+4b &> b^2.
\end{aligned}$$

Thus, $F_2(y)$ is increasing in y. Therefore, it is sufficient to show that $F_2(-(1-\tilde{c}_1)(1-b)) > 0$ because

$-(1-\tilde{c}_1)(1-b) < y_L$ and $F_2(y)$ is increasing in y.

$$
\begin{aligned}
F_2(-(1-\tilde{c}_1)(1-b)) &> 0 \\
-(1-\tilde{c}_1)(1-b)\left(b - 8b\frac{(1-b)(1+b)}{(b^2-4)^2}(2-b^2)\right) & \\
+(1-\tilde{c}_1)\left((2-b-b^2) - 8\frac{(1-b)(1+b)(2-b)}{(b^2-4)^2}(2-b^2)\right) &> 0 \\
(1-b)\left(8b\frac{(1-b)(1+b)}{(b^2-4)^2}(2-b^2) - b\right) & \\
+\left((2+b)(1-b) - 8\frac{(1-b)(1+b)(2-b)}{(b^2-4)^2}(2-b^2)\right) &> 0 \\
\left(8b\frac{(1-b)(1+b)}{(b^2-4)^2}(2-b^2) - b\right) + \left((2+b) - 8\frac{(1+b)(2-b)}{(b^2-4)^2}(2-b^2)\right) &> 0 \\
8b\frac{(1-b)(1+b)}{(b^2-4)^2}(2-b^2) + 2 - 8\frac{(1+b)(2-b)}{(b^2-4)^2}(2-b^2) &> 0 \\
2 + 8\frac{(1-b)(2-b^2)}{(b^2-4)^2}\left(b^2+b-(2-b)\right) &> 0 \\
2 + 8\frac{(1-b)(2-b^2)}{(b^2-4)^2}\left(-2+2b+b^2\right) &> 0 \\
2 - 8\frac{(1-b)(2-b^2)}{(b^2-4)^2}(2+b)(1-b) &> 0 \\
2\left(b^2-4\right)^2 &> 8(1-b)\left(2-b^2\right)(2+b)(1-b) \\
\left(b^2-4\right)^2 &> 4(1-b)\left(2-b^2\right)(2+b)(1-b) \\
b^4 - 8b^2 + 16 &> -4b^5 + 20b^3 - 8b^2 - 24b + 16 \\
24b + b^4 + 4b^5 &> 20b^3 \\
24 + b^3 + 4b^4 &> 20b^2
\end{aligned}
$$

which holds for all b. □

B.5.3 Proof of Lemma 2 (Quantity Competition)

Lemma B.7. *Suppose $x - a \neq 0$. Then, $\delta^*(\tilde{c}_1, x, a)$ is increasing in x when $x > a$ and decreasing in x when $x < a$, under differentiated product quantity competition.*

Proof. There are two cases to consider:

Case 1 ($x - a > 0$): By Lemma B.4, the industry critical discount factor is manager 2's critical discount factor:

$$
\delta_2(\tilde{c}_1, x, a) = \frac{b^2\left((1-\tilde{c}_1)(1-b) + bx\right)^2}{\left((2-b-b^2)(1-\tilde{c}_1) - x(2-b^2)\right)^2 - \frac{16(1-b^2)^2}{(b^2-4)^2}\left((2-b)(1-\tilde{c}_1) - 2x\right)^2}.
$$

$\delta_2(\tilde{c}_1, x, a)$ is increasing in x by Lemma B.2.

Case 2 ($x - a < 0$): By Lemma B.4, the industry critical discount factor is manager 1's critical discount factor:

$$
\delta_1(\tilde{c}_1, x, a) = \frac{b^2\left((1-\tilde{c}_1)(1-b) - (x-a)\right)^2}{\left((2-b-b^2)(1-\tilde{c}_1) + b(x-a)\right)^2 - 16(b^2-1)^2\left(\frac{((2-b)(1-\tilde{c}_1)+b(x-a))^2}{(b^2-4)^2}\right)}.
$$

$\delta_1(\tilde{c}_1, x, a)$ is decreasing in x by Lemma B.3. □

B.5.4 Proof of Proposition 2 (Quantity Competition)

Proposition. *Suppose* $c_1 = c_2$ *and* $a = 0$*. Then, (i)* $\delta^*(\theta_1, \theta_2) = \delta^*(0,0)$ *if* $\theta_1 = \theta_2$*, and (ii)* $\delta^*(\theta_1, \theta_2) > \delta^*(0,0)$ *if* $\theta_1 \neq \theta_2$*.*

Proof. Part i) $\theta_1 = \theta_2$, $c_1 = c_2$, and $a = 0$ imply that $x - a = 0$. Substituting $x - a = 0$ into $\delta^*(\theta_1, \theta_2)$ (see Subsection (A.3)) yields

$$\delta^*_{sym} = \frac{(2+b)^2}{b^2 + 8b + 8}.$$

Additionally, substituting $c_1 = c_2$, $a = 0$ and $\theta_1 = \theta_2 = 0$ into $\delta^*(\theta_1, \theta_2)$ yields $\delta(0,0) = \delta^*_{sym}$. Therefore, $\delta^*(\theta_1, \theta_2) = \delta^*(0,0)$ when $\theta_1 = \theta_2$.

Part ii) $\theta_1 \neq \theta_2$, $c_1 = c_2$ and $a = 0$ imply that $x > a$ or $x < a$. First, consider the case of $x > a$. By Lemma 2, $\delta^*(\theta_1, \theta_2)$ is increasing in x for all $x > a$ and, thus, $\delta^*(\theta_1, \theta_2) > \delta^*_{sym}$ when $x - a > 0$. Next, consider the case of $x < a$. By Lemma 2, $\delta^*(\theta_1, \theta_2)$ is decreasing in x for all $x < a$ and, thus, $\delta^*(\theta_1, \theta_2) > \delta^*_{sym}$ when $x < a$. As $\delta^*(0,0) = \delta^*_{sym}$ (see part i), it follows that $\delta^*(\theta_1, \theta_2) > \delta^*(0,0)$ if $\theta_1 \neq \theta_2$. □

B.5.5 Proof of Proposition 3 (Quantity Competition)

Proposition. $\delta^*(\theta_1, \theta_2) < \delta^*(0,0)$ *if* $|(1-\theta_2)\, c_2 - (1-\theta_1)\, c_1 - a| < |c_2 - c_1 - a|$ *(Condition 1).*

Proof. Note that $\delta^*(A, B, C) = \delta^*(A, B - C, 0)$ (see Section B.3). Thus, $\delta^*(\tilde{c}_1, x, a) = \delta^*(\tilde{c}_1, x - a, 0)$. Let $z = c_2 - c_1 - a$. When $\theta_1 = \theta_2 = 0$, the critical discount factor is $\delta^*(c_1, c_2 - c_1, a) = \delta^*(c_1, z, 0)$. We wish to show $\delta^*(\tilde{c}_1, x - a, 0) < \delta^*(c_1, z, 0)$ when Condition 1 holds. Suppose Condition 1 holds (i.e., $|x - a| < |z|$). There are four cases to consider.

Case 1 $z > 0$ *and* $x - a \geq 0$*:* The result follows from the observation that $\delta^*(c_1, z, 0) \geq \delta^*(\tilde{c}_1, z, 0) > \delta^*(\tilde{c}_1, x - a, 0)$ where the first inequality follows from Lemma 1 (and $\tilde{c}_1 \leq c_1$) and the second inequality follows from Lemma 2 (and $0 \leq x - a < z$).

Case 2 $z > 0$ *and* $x - a < 0$*:* Note that $\delta^*(A, B, 0) = \delta^*(A + B, -B, 0)$ for any A and B by symmetry. Thus, $\delta^*(c_1, z, 0) = \delta^*(c_1 + z, -z, 0)$ holds. The result follows from

$$\delta^*(c_1, z, 0) = \delta^*(c_1 + z, -z, 0) > \delta^*(c_1 + z, x - a, 0) > \delta^*(\tilde{c}_1, x - a, 0)$$

where the first inequality follows from $-z < x - a < 0$ (by $|x - a| < |z|$) and Lemma 2. The second inequality follows from $c_1 + z > \tilde{c}_1$ and Lemma 1.

Case 3 $z < 0$ *and* $x - a > 0$*:* The result follows from

$$\delta^*(c_1, z, 0) = \delta^*(c_1 + z, -z, 0) > \delta^*(c_1 + z, x - a, 0) > \delta^*(\tilde{c}_1, x - a, 0)$$

where the first inequality follows from $0 < x - a < -z$ (by $|x - a| < |z|$) and Lemma 2. The second inequality follows from Lemma 1 and $c_1 + z > \tilde{c}_1$ which holds because

$$\tilde{c}_1 = \tilde{c}_2 - x < \tilde{c}_2 - a \leq c_2 - a = c_1 + c_2 - c_1 - a = c_1 + z$$

where the first inequality follows from $x > a$, and the second inequality follows from $(1 - \theta_2)c_2 \leq c_2$.

Case 4 $z < 0$ *and* $x - a \leq 0$*:* The result follows from $\delta^*(c_1, z, 0) \geq \delta^*(\tilde{c}_1, z, 0) > \delta^*(\tilde{c}_1, x - a, 0)$ where the first inequality follows from Lemma 1 (and $\tilde{c}_1 \leq c_1$) and the second inequality follows from Lemma 2 (and $z < x - a \leq 0$). □

B.5.6 Proof of Proposition 4 (Quantity Competition)

Let

$$\pi_i(\theta_1, \theta_2) = \begin{cases} \pi_i^C(\theta_1, \theta_2) & \text{if } \delta \geq \delta^*(\theta_1, \theta_2) \\ \pi_i^N(\theta_1, \theta_2) & \text{if } \delta < \delta^*(\theta_1, \theta_2) \end{cases}$$

denote owner i's profits as in the main text.

Proposition. *$\pi_1(\theta_1, \theta_2) > \pi_1(0,0)$ and $\pi_2(\theta_1, \theta_2) > \pi_2(0,0)$ if $1 - c_1 > \theta_1 c_1,\ 1 - c_2 + a > \theta_2 c_2$,*

$$\theta_1 < \frac{(1 - \tilde{c}_2 + a) - b(1 - c_1)}{bc_1} + 4\frac{\left(\left(2 - b^2\right)(1 - c_2 + a) - b(1 - c_1)\right)^2}{b(4 - b^2)^2(2c_2 - \tilde{c}_2 - a - 1)c_1},$$

$$\theta_2 < \frac{(1 - \tilde{c}_1) - b(1 - c_2 + a)}{bc_2} + 4\frac{\left(\left(2 - b^2\right)(1 - c_1) - b(1 - c_2 + a)\right)^2}{b(4 - b^2)^2(2c_1 - \tilde{c}_1 - 1)c_2}$$

and $\delta^*(\theta_1, \theta_2) \leq \delta < \delta^*(0,0) < 1$.

Proof. $\delta^*(\theta_1, \theta_2) \leq \delta < \delta^*(0,0)$ ensures that collusion is sustainable under sales-based compensation and unsustainable under profit-based compensation. Thus, $\pi_i(\theta_1, \theta_2) = \pi_i^C(\theta_1, \theta_2)$ and $\pi_i(0,0) = \pi_i^N(0,0)$ for $i = 1,2$. It remains to show that $\pi_1^C(\theta_1, \theta_2) > \pi_1^N(0,0)$ and $\pi_2^C(\theta_1, \theta_2) > \pi_2^N(0,0)$ for θ_1 and θ_2 satisfying the assumptions in the proposition.

$\delta^*(0,0) < 1$ ensures both firms are active in the Nash equilibrium (see Subsection B.3 and Subsection A.2.1). When managers collude and sales-based compensation weights are θ_1 and θ_2, $q_1^C = \frac{1-\tilde{c}_1-b(1-\tilde{c}_2+a)}{2(1-b^2)}$ and $q_2^C = \frac{1-\tilde{c}_2+a-b(1-\tilde{c}_1)}{2(1-b^2)}$. Owner 1's profits when managers collude and compensation is sales-based are

$$\begin{aligned}
\pi_1^C(\theta_1, \theta_2) &= \left(P_1\left(q_1^C, q_2^C\right) - c_1\right) q_1^C \\
&= \left(1 - q_1^C - bq_2^C - c_1\right) q_1^C \\
&= \frac{1 - \tilde{c}_1 - b(1 - \tilde{c}_2 + a)}{2(1 - b^2)} \left(1 - \frac{1 - \tilde{c}_1 - b(1 - \tilde{c}_2 + a)}{2(1 - b^2)} - b\frac{1 - \tilde{c}_2 + a - b(1 - \tilde{c}_1)}{2(1 - b^2)} - c_1\right) \\
&= \frac{1 - \tilde{c}_1 - b(1 - \tilde{c}_2 + a)}{4(1 - b^2)^2} \left(2\left(1 - b^2\right) - (1 - \tilde{c}_1 - b(1 - \tilde{c}_2 + a)) - b(1 - \tilde{c}_2 + a - b(1 - \tilde{c}_1)) - 2c_1\left(1 - b^2\right)\right) \\
&= \frac{1 - \tilde{c}_1 - b(1 - \tilde{c}_2 + a)}{4(1 - b^2)^2} \left(2 - 2b^2 - 1 + \tilde{c}_1 + b - b\tilde{c}_2 + ba - b + b\tilde{c}_2 - ba + b^2 - b^2\tilde{c}_1 - 2c_1\left(1 - b^2\right)\right) \\
&= \frac{1 - \tilde{c}_1 - b(1 - \tilde{c}_2 + a)}{4(1 - b^2)^2} \left(1 - b^2 + \tilde{c}_1 - b^2\tilde{c}_1 - 2c_1\left(1 - b^2\right)\right) \\
&= \frac{1 - \tilde{c}_1 - b(1 - \tilde{c}_2 + a)}{4(1 - b^2)^2} \left(1 - b^2 + \tilde{c}_1\left(1 - b^2\right) - 2c_1\left(1 - b^2\right)\right) \\
&= \frac{1 - \tilde{c}_1 - b(1 - \tilde{c}_2 + a)}{4(1 - b^2)} (1 + \tilde{c}_1 - 2c_1).
\end{aligned}$$

Owner 2's profits when managers collude and compensation is sales-based are

$$\begin{aligned}
\pi_2^C(\theta_1, \theta_2) &= \left(P_2\left(q_1^C, q_2^C\right) - c_2\right) q_2^C \\
&= \left(1 - q_2^C - bq_1^C - c_1\right) q_2^C \\
&= \frac{1 - \tilde{c}_2 + a - b(1 - \tilde{c}_1)}{2(1 - b^2)} \left(1 + a - \frac{1 - \tilde{c}_2 + a - b(1 - \tilde{c}_1)}{2(1 - b^2)} - b\frac{1 - \tilde{c}_1 - b(1 - \tilde{c}_2 + a)}{2(1 - b^2)} - c_2\right) \\
&= \frac{1 - \tilde{c}_2 + a - b(1 - \tilde{c}_1)}{4(1 - b^2)^2} \left(2\left(1 - b^2\right)(1 + a - c_2) - (1 - \tilde{c}_2 + a - b(1 - \tilde{c}_1)) - b(1 - \tilde{c}_1 - b(1 - \tilde{c}_2 + a))\right) \\
&= \frac{1 - \tilde{c}_2 + a - b(1 - \tilde{c}_1)}{4(1 - b^2)^2} \left(2 - 2b^2 + 2a - 2ab^2 - 1 + \tilde{c}_2 - a + b - b\tilde{c}_1 - b + b\tilde{c}_1 + b^2 - b^2\tilde{c}_2 + b^2 a - 2c_2\left(1 - b^2\right)\right) \\
&= \frac{1 - \tilde{c}_2 + a - b(1 - \tilde{c}_1)}{4(1 - b^2)^2} \left(1 - b^2 + a - ab^2 + \tilde{c}_2 - b^2\tilde{c}_2 - 2c_2\left(1 - b^2\right)\right) \\
&= \frac{1 - \tilde{c}_2 + a - b(1 - \tilde{c}_1)}{4(1 - b^2)^2} \left(1 - b^2 + \tilde{c}_2\left(1 - b^2\right) - 2c_2\left(1 - b^2\right) + a\left(1 - b^2\right)\right) \\
&= \frac{1 - \tilde{c}_2 + a - b(1 - \tilde{c}_1)}{4(1 - b^2)} (1 + \tilde{c}_2 - 2c_2 + a).
\end{aligned}$$

Nash equilibrium quantities when managers compete and compensation is strictly profit-based (i.e., $\theta_1 = \theta_2 = 0$) are

$$q_1^N = \frac{2 - b - 2c_1 - ab + bc_2}{4 - b^2} \tag{24}$$

and

$$q_2^N = \frac{2 - b - 2c_2 + 2a + bc_1}{4 - b^2}. \tag{25}$$

Thus, firm profits when managers compete and compensation is strictly profit-based are

$$\begin{aligned} \pi_1^N(0,0) &= \left(P_1\left(q_1^N, q_2^N\right) - c_1\right) q_1^N \\ &= \frac{\left(2\left(1-c_1\right) - b\left(1 - c_2 + a\right)\right)^2}{\left(4 - b^2\right)^2} \end{aligned}$$

and

$$\begin{aligned} \pi_2^N(0,0) &= \left(P_2\left(q_1^N, q_2^N\right) - c_2\right) q_2^N \\ &= \frac{\left(2\left(1-c_2+a\right) - b\left(1 - c_1\right)\right)^2}{\left(4 - b^2\right)^2}. \end{aligned}$$

For Firm 1, it follows that

$$\begin{aligned} &\pi_1^C(\theta_1,\theta_2) > \pi_1^N(0,0) \\ \iff & \frac{1 - \tilde{c}_1 - b\left(1 - \tilde{c}_2 + a\right)}{4\left(1-b^2\right)}\left(1 + \tilde{c}_1 - 2c_1\right) > \frac{\left(2\left(1-c_1\right) - b\left(1 - c_2 + a\right)\right)^2}{\left(4 - b^2\right)^2}. \end{aligned}$$

Thus, $\pi_1^C(\theta_1,\theta_2) > \pi_1^N(0,0)$ holds if and only if (note that $1 - c_1 > \theta_1 c_1$ (which is assumed in the Proposition) $\implies 1 - 2c_1 + \tilde{c}_1 > 0$)

$$\begin{aligned} &\iff \tilde{c}_2 > \frac{b + \tilde{c}_1 + ab - 1}{b} + 4\left(1-b^2\right)\frac{\left(2\left(1-c_1\right) - b\left(1 - c_2 + a\right)\right)^2}{b\left(4-b^2\right)^2\left(1 - 2c_1 + \tilde{c}_1\right)} \\ &\iff c_2\left(1-\theta_2\right) > \frac{b + \tilde{c}_1 + ab - 1}{b} + 4\left(1-b^2\right)\frac{\left(2\left(1-c_1\right) - b\left(1 - c_2 + a\right)\right)^2}{b\left(4-b^2\right)^2\left(1 - 2c_1 + \tilde{c}_1\right)} \\ &\iff -c_2\theta_2 > \frac{b + \tilde{c}_1 + ab - 1 - bc_2}{b} + 4\left(1-b^2\right)\frac{\left(2\left(1-c_1\right) - b\left(1 - c_2 + a\right)\right)^2}{b\left(4-b^2\right)^2\left(1 - 2c_1 + \tilde{c}_1\right)} \\ &\iff -c_2\theta_2 > \frac{b\left(1 - c_2 + a\right) - \left(1 - \tilde{c}_1\right)}{b} + 4\left(1-b^2\right)\frac{\left(2\left(1-c_1\right) - b\left(1 - c_2 + a\right)\right)^2}{b\left(4-b^2\right)^2\left(1 - 2c_1 + \tilde{c}_1\right)} \\ &\iff \theta_2 < \frac{\left(1 - \tilde{c}_1\right) - b\left(1 - c_2 + a\right)}{bc_2} + 4\left(1-b^2\right)\frac{\left(2\left(1-c_1\right) - b\left(1 - c_2 + a\right)\right)^2}{b\left(4-b^2\right)^2\left(2c_1 - \tilde{c}_1 - 1\right)c_2} \end{aligned} \tag{26}$$

which holds. For firm 2, it follows that

$$\begin{aligned} &\pi_2^C(\theta_1,\theta_2) > \pi_2^N(0,0) \\ \iff & \frac{1 - \tilde{c}_2 + a - b\left(1 - \tilde{c}_1\right)}{4\left(1-b^2\right)}\left(1 + \tilde{c}_2 - 2c_2 + a\right) > \frac{\left(2\left(1-c_2+a\right) - b\left(1 - c_1\right)\right)^2}{\left(4 - b^2\right)^2} \end{aligned}$$

which holds if (note that $1 - c_2 + a > \theta_2 c_2$ (which is assumed in the Proposition) $\implies 1 + \tilde{c}_2 - 2c_2 + a > 0$)

$$\begin{aligned}
\tilde{c}_1 &> \frac{b + \tilde{c}_2 - a - 1}{b} + 4\left(1 - b^2\right) \frac{\left(2\left(1 - c_2 + a\right) - b\left(1 - c_1\right)\right)^2}{b\left(4 - b^2\right)^2 \left(1 + \tilde{c}_2 - 2c_2 + a\right)} \\
c_1\left(1 - \theta_1\right) &> \frac{b + \tilde{c}_2 - a - 1}{b} + 4\left(1 - b^2\right) \frac{\left(2\left(1 - c_2 + a\right) - b\left(1 - c_1\right)\right)^2}{b\left(4 - b^2\right)^2 \left(1 + \tilde{c}_2 - 2c_2 + a\right)} \\
-c_1\theta_1 &> \frac{b + \tilde{c}_2 - a - 1 - bc_1}{b} + 4\left(1 - b^2\right) \frac{\left(2\left(1 - c_2 + a\right) - b\left(1 - c_1\right)\right)^2}{b\left(4 - b^2\right)^2 \left(1 + \tilde{c}_2 - 2c_2 + a\right)} \\
-c_1\theta_1 &> \frac{b\left(1 - c_1\right) - \left(1 - \tilde{c}_2 + a\right)}{b} + 4\left(1 - b^2\right) \frac{\left(2\left(1 - c_2 + a\right) - b\left(1 - c_1\right)\right)^2}{b\left(4 - b^2\right)^2 \left(1 - 2c_2 + \tilde{c}_2 + a\right)} \\
\theta_1 &< \frac{\left(1 - \tilde{c}_2 + a\right) - b\left(1 - c_1\right)}{bc_1} + 4\left(1 - b^2\right) \frac{\left(2\left(1 - c_2 + a\right) - b\left(1 - c_1\right)\right)^2}{b\left(4 - b^2\right)^2 \left(2c_2 - 1 - a - \tilde{c}_2\right) c_1} \qquad (27)
\end{aligned}$$

which holds. □

B.5.7 Proof of Proposition 5 (Quantity Competition)

Let $P_i^C(\theta_1, \theta_2)$ denote product i's inverse demand when managers collude and sales weights are θ_1 and θ_2. Let $P_i^N(\theta_1, \theta_2)$ denote product i's inverse demand when managers compete and sales weights are θ_1 and θ_2. Quantities under collusion are denoted $q_1^C(\theta_1, \theta_2)$ and $q_2^C(\theta_1, \theta_2)$ while quantities under competition are denoted $q_1^N(\theta_1, \theta_2)$ and $q_2^N(\theta_1, \theta_2)$.

$$\begin{aligned}
CS^j(\theta_1, \theta_2) = {} & q_1^j(\theta_1, \theta_2) + (1 + a)\, q_2^j(\theta_1, \theta_2) - \frac{1}{2}\left(q_1^j(\theta_1, \theta_2)^2 + q_2^j(\theta_1, \theta_2)^2 + 2b q_1^j(\theta_1, \theta_2) q_2^j(\theta_1, \theta_2)\right) \\
& - P_1^j(\theta_1, \theta_2) q_1^j(\theta_1, \theta_2) - P_2^j(\theta_1, \theta) q_2^j(\theta_1, \theta_2) + I
\end{aligned}$$

denotes consumer surplus when managers collude ($j = C$) or compete ($j = N$) and sales weights are θ_1 and θ_2. I denotes income.

Proposition. $CS^C(\theta_1, \theta_2) < CS^N(0, 0)$ *if* $\theta_1 < \frac{1}{c_1}\left(\frac{2b(1-c_2+a)-b^2(1-c_1)}{4-b^2}\right)$, $\theta_2 < \frac{1}{c_2}\left(\frac{2b(1-c_1)-b^2(1-c_2+a)}{4-b^2}\right)$ *and* $\delta^*(\theta_1, \theta_2) \leq \delta < \delta^*(0, 0) < 1$.

Proof. $\delta^*(\theta_1, \theta_2) \leq \delta < \delta^*(0, 0)$ ensures that collusion is sustainable under sales-based compensation and unsustainable under profit-based compensation. Thus, $CS(\theta_1, \theta_2) = CS^C(\theta_1, \theta_2)$ and $CS(0, 0) = CS^N(0, 0)$ for $i = 1, 2$. It remains to show that $CS^C(\theta_1, \theta_2) < CS^N(0, 0)$ for θ_1 and θ_2 satisfying the assumptions in the proposition.

$\delta^*(0, 0) < 1$ ensures both firms are active in the Nash equilibrium (see Subsection B.3 and Subsection A.2.1). Prices under quantity competition if managers compete and compensation is sales-based are

$$\begin{aligned}
p_1^N(0, 0) &= P_1(q_1^N, q_2^N) \\
&= 1 - q_1^N - b q_2^N \\
&= 1 - \frac{2 - b - 2c_1 - ab + bc_2}{4 - b^2} - b\frac{2 - b - 2c_2 + 2a + bc_1}{4 - b^2} \\
&= \frac{1}{4 - b^2}\left[4 - b^2 - \left(2 - b - 2c_1 - ab + bc_2\right) - b\left(2 - b - 2c_2 + 2a + bc_1\right)\right] \\
&= \frac{1}{4 - b^2}\left[4 - b^2 - 2 + b + 2c_1 + ab - bc_2 - 2b + b^2 + 2bc_2 - 2ab - b^2 c_1\right] \\
&= \frac{1}{4 - b^2}\left[2 - b + 2c_1 - ab + bc_2 - b^2 c_1\right] \\
&= \frac{1}{4 - b^2}\left[2 - b\left(1 - c_2 + a\right) + \left(2 - b^2\right) c_1\right]
\end{aligned}$$

and

$$
\begin{aligned}
p_2^N(0,0) &= P_2(q_1^N, q_2^N) \\
&= 1 - q_2^N - bq_1^N \\
&= 1 + a - \frac{2 - b - 2c_2 + 2a + bc_1}{4 - b^2} - b\frac{2 - b - 2c_1 - ab + bc_2}{4 - b^2} \\
&= \frac{1}{4 - b^2}\left[4 - b^2 + 4a - ab^2 - (2 - b - 2c_2 + 2a + bc_1) - b\,(2 - b - 2c_1 - ab + bc_2)\right] \\
&= \frac{1}{4 - b^2}\left[4 - b^2 + 4a - ab^2 - 2 + b + 2c_2 - 2a - bc_1 - 2b + b^2 + 2bc_1 + ab^2 - b^2 c_2\right] \\
&= \frac{1}{4 - b^2}\left[2 - b + 2c_2 + 2a + bc_1 - b^2 c_2\right] \\
&= \frac{1}{4 - b^2}\left[2 - b\,(1 - c_1) + \left(2 - b^2\right) c_2 + 2a\right].
\end{aligned}
$$

Prices when managers collude and compensation is sales-based are

$$
\begin{aligned}
p_1^C(\theta_1, \theta_2) &= P_1\left(q_1^C, q_2^C\right) \\
&= 1 - q_1^C - bq_2^C \\
&= 1 - \frac{1 - \tilde{c}_1 - b\,(1 - \tilde{c}_2 + a)}{2\,(1 - b^2)} - b\frac{1 - \tilde{c}_2 + a - b\,(1 - \tilde{c}_1)}{2\,(1 - b^2)} \\
&= \frac{1}{2\,(1 - b^2)}\left(2\left(1 - b^2\right) - (1 - \tilde{c}_1 - b\,(1 - \tilde{c}_2 + a)) - b\,(1 - \tilde{c}_2 + a - b\,(1 - \tilde{c}_1))\right) \\
&= \frac{1}{2\,(1 - b^2)}\left(2 - 2b^2 - 1 + \tilde{c}_1 + b - b\tilde{c}_2 + ba - b + b\tilde{c}_2 - ba + b^2 - b^2\tilde{c}_1\right) \\
&= \frac{1}{2\,(1 - b^2)}\left(1 - b^2 + \tilde{c}_1 - b^2\tilde{c}_1\right) \\
&= \frac{1}{2\,(1 - b^2)}\left(1 - b^2 + \tilde{c}_1\left(1 - b^2\right)\right) \\
&= \frac{1 + \tilde{c}_1}{2}.
\end{aligned}
$$

and

$$
\begin{aligned}
p_2^C(\theta_1, \theta_2) &= P_2\left(q_1^C, q_2^C\right) \\
&= 1 - q_2^C - bq_1^C \\
&= 1 + a - \frac{1 - \tilde{c}_2 + a - b\,(1 - \tilde{c}_1)}{2\,(1 - b^2)} - b\frac{1 - \tilde{c}_1 - b\,(1 - \tilde{c}_2 + a)}{2\,(1 - b^2)} \\
&= \frac{1}{2\,(1 - b^2)}\left(2\left(1 - b^2\right) + 2a\left(1 - b^2\right) - (1 - \tilde{c}_2 + a - b\,(1 - \tilde{c}_1)) - b\,(1 - \tilde{c}_1 - b\,(1 - \tilde{c}_2 + a))\right) \\
&= \frac{1}{2\,(1 - b^2)}\left(2 - 2b^2 + 2a - 2ab^2 - 1 + \tilde{c}_2 - a + b - b\tilde{c}_1 - b + b\tilde{c}_1 + b^2 - b^2\tilde{c}_2 + b^2 a\right) \\
&= \frac{1}{2\,(1 - b^2)}\left(1 - b^2 + a - ab^2 + \tilde{c}_2 - b^2\tilde{c}_2\right) \\
&= \frac{1}{2\,(1 - b^2)}\left(1 - b^2 + \tilde{c}_2\left(1 - b^2\right) + a\left(1 - b^2\right)\right) \\
&= \frac{1 + \tilde{c}_2 + a}{2}.
\end{aligned}
$$

Note that $CS^C(\theta_1,\theta_2) < CS^N(0,0)$ if

$$\begin{aligned}
p_1^N(0,0) &< p_1^C(\theta_1,\theta_2) \\
\frac{2-b+2c_1-ab+bc_2-c_1b^2}{4-b^2} &< \frac{1+\tilde{c}_1}{2} \\
\frac{2-b+2c_1-ab+bc_2-c_1b^2}{4-b^2} &< \frac{1+(1-\theta_1)\,c_1}{2} \\
\theta_1 &< \frac{1}{c_1}\left(1+c_1-2\frac{2-b+2c_1-ab+bc_2-c_1b^2}{4-b^2}\right) \\
\theta_1 &< \frac{1}{c_1}\left(1+c_1-\frac{4-2b+4c_1-2ab+2bc_2-2c_1b^2}{4-b^2}\right) \\
\theta_1 &< \frac{1}{c_1}\left(\frac{4-b^2+4c_1-c_1b^2-4+2b-4c_1+2ab-2bc_2+2c_1b^2}{4-b^2}\right) \\
\theta_1 &< \frac{1}{c_1}\left(\frac{-b^2+2b+2ab-2bc_2+c_1b^2}{4-b^2}\right) \\
\theta_1 &< \frac{1}{c_1}\left(\frac{2b\,(1-c_2+a)-b^2\,(1-c_1)}{4-b^2}\right)
\end{aligned}$$

and

$$\begin{aligned}
p_2^N(0,0) &< p_2^C(\theta_1,\theta_2) \\
\frac{2-b+bc_1+2c_2-b^2c_2+2a}{4-b^2} &< \frac{1+\tilde{c}_2+a}{2} \\
\frac{2-b+bc_1+2c_2-b^2c_2+2a}{4-b^2} &< \frac{1+(1-\theta_2)\,c_2+a}{2} \\
\theta_2 &< \frac{1}{c_2}\left(1+c_2+a-2\frac{2-b+bc_1+2c_2-b^2c_2+2a}{4-b^2}\right) \\
\theta_2 &< \frac{1}{c_2}\left(1+c_2+a-\frac{4-2b+2bc_1+4c_2-2b^2c_2+4a}{4-b^2}\right) \\
\theta_2 &< \frac{1}{c_2}\left(1+c_2+a+\frac{-4+2b-2bc_1-4c_2+2b^2c_2-4a}{4-b^2}\right) \\
\theta_2 &< \frac{1}{c_2}\left(\frac{4-b^2+4c_2-b^2c_2+4a-b^2a-4+2b-2bc_1-4c_2+2b^2c_2-4a}{4-b^2}\right) \\
\theta_2 &< \frac{1}{c_2}\left(\frac{-b^2+b^2c_2-b^2a+2b-2bc_1}{4-b^2}\right) \\
\theta_2 &< \frac{1}{c_2}\left(\frac{2b\,(1-c_1)-b^2(1-c_2+a)}{4-b^2}\right).
\end{aligned}$$

□

B.6 Sales Weights Facilitating Collusion, Enhancing Firm Profit and Reducing Consumer Surplus

In this subsection, we provide a sufficient condition which ensures that there exists sales weights θ_1 and θ_2 which facilitate collusion, enhance firm profit and reduce consumer surplus. Note that, by Lemma B.5, $\delta^*(0,0) < 1$ if and only if

$$(1-c_1)\left(1+\frac{\left(8b+b^2\sqrt{b^2+8}+b^3-4\sqrt{b^2+8}\right)}{8\,(1-b^2)}\right) < c_2-c_1-a < (1-c_1)\left(1+\frac{\left(-8b-b^3+b^2\sqrt{b^2+8}-4\sqrt{b^2+8}\right)}{2\,(b^2+8)}\right).$$

Thus, $\delta^*(0,0) < 1$ if and only if the asymmetry between firms is sufficiently moderate.

Proposition B.1. *Suppose $c_2 - a \neq c_1$ and $\delta^*(0,0) < 1$. Then there exist $\theta_1, \theta_2 \in [0,1]$ such that*
i) $\delta^*(\theta_1,\theta_2) < \delta^*(0,0)$,
ii) $\pi_1^C(\theta_1,\theta_2) > \pi_1^N(0,0)$ and $\pi_2^C(\theta_1,\theta_2) > \pi_2^N(0,0)$, and
iii) $CS^C(\theta_1,\theta_2) < CS^N(0,0)$.

Proof. The proof proceeds by constructing a (θ_1,θ_2) such that part (i), (ii) and (iii) hold.[15] Without loss of generality, assume $c_2 - a > c_1$. Let $\theta_1 = 0$ and let $\theta_2 = \epsilon$ where $\epsilon > 0$ is small. Note that Condition 1 holds by

$$|c_2(1-\theta_2) - a - c_1(1-\theta_1)| = |c_2(1-\epsilon) - a - c_1| < c_2 - a - c_1$$

for sufficiently small ϵ. Thus, part (i) holds.

Next, consider part (ii). First, note that $\delta^*(0,0) < 1$ implies both firms are active in the Nash equilibrium when $\theta_1 = \theta_2 = 0$. Additionally, $\delta^*(0,0) < 1$ implies $M_1^N(0,0) < M_1^C(0,0)$ and $M_2^N(0,0) < M_2^C(0,0)$ (see the proof of Lemma B.5). This implies $M_1^N(0,0) = \pi_1^N(0,0) < \pi_1^C(0,0) < M_1^C(0,0)$ and $M_2^N(0,0) = \pi_2^N(0,0) < \pi_2^C(0,0) < M_2^C(0,0)$. By continuity of $\pi_2^C(\theta_1,\theta_2)$, $\pi_2^C(0,0) > \pi_2^N(0,0)$ implies $\pi_2^C(\theta_1,\theta_2) = \pi_2^C(0,\epsilon) > \pi_2^N(0,0)$ for sufficiently small ϵ. Thus, $\pi_1^C(\theta_1,\theta_2) = \pi_1^C(0,0) > \pi_1^N(0,0)$ and $\pi_2^C(\theta_1,\theta_2) = \pi_2^C(0,\epsilon) > \pi_2^N(0,0)$ and part (ii) holds for both firms.

Next, consider Part (iii). We show that the conditions of Proposition 5 hold for $\theta_1 = \theta_2 = 0$ which implies, by continuity, that the inequalities are also satisfied for $\theta_1 = 0$ and $\theta_2 = \epsilon$. First, note that

$$\begin{aligned}
\theta_1 = 0 <& \frac{1}{c_1}\left(\frac{2b(1-c_2+a) - b^2(1-c_1)}{4-b^2}\right) \\
0 <& \, 2b(1-c_2+a) - b^2(1-c_1) \\
b^2(1-c_1) <& \, 2b(1-c_2+a) \\
\frac{b}{2}(1-c_1) <& \, 1 - c_2 + a \\
\frac{b}{2}(1-c_1) <& \, 1 - c_1 - x + a \\
x - a <& \left(1 - \frac{b}{2}\right)(1-c_1).
\end{aligned}$$

To see that this inequality holds, note that $\delta^*(0,0) < 1$ implies (by Lemma B.5 and Lemma B.1) that

$$x - a < (1-b)(1-c_1)$$

which implies

$$x - a < \left(1 - \frac{b}{2}\right)(1-c_1)$$

holds. Second, note that

$$\begin{aligned}
0 <& \frac{1}{c_2}\left(\frac{2b(1-c_1) - b^2(1-c_2+a)}{4-b^2}\right) \\
0 <& \, 2b(1-c_1) - b^2(1-c_2+a) \\
b^2(1-c_2+a) <& \, 2b(1-c_1) \\
1 - c_1 - x + a <& \, \frac{2}{b}(1-c_1) \\
(1-c_1) - \frac{2}{b}(1-c_1) <& \, x - a \\
\left(1 - \frac{2}{b}\right)(1-c_1) <& \, x - a.
\end{aligned}$$

[15] The set of (θ_1,θ_2) that satisfy conditions i, ii and iii is not, in general, a singleton and can include a wide range of (θ_1,θ_2) values.

To see that this inequality holds, note that $\delta^*(0,0) < 1$ implies (by Lemma B.5 and Lemma B.1)

$$\left(1 - \frac{2-b^2}{b}\right)(1-c_1) < x - a$$

or

$$\left(1 - \frac{2}{b} + b\right)(1-c_1) < x - a.$$

Additionally,

$$\left(1 - \frac{2}{b} + b\right)(1-c_1) > \left(1 - \frac{2}{b}\right)(1-c_1).$$

Therefore, $\left(1 - \frac{2}{b} + b\right)(1-c_1) < x - a$ implies that

$$\left(1 - \frac{2}{b}\right)(1-c_1) < x - a.$$

Thus, the conditions of Proposition 5 are satisfied for $\theta_1 = \theta_2 = 0$ and, by continuity, they are also satisfied for $\theta_1 = 0$ and $\theta_2 = \epsilon$ for sufficiently small ϵ. Thus, Part (iii) holds. □

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