Technical Appendix to Accompany

"On the Design of Price Caps as Sanctions"

by D. Sappington and D. Turner

Equations from the Text

$$C^R(q_A, q_N) \;=\; c_A\, q_A + \frac{k_A}{2}\,[\,q_A\,]^2 + c_N\, q_N + \frac{k_N}{2}\,[\,q_N\,]^2 + \frac{k^R}{2}\,[\,q_A + q_N\,]^2 \,. \tag{1}$$

$$D \;\equiv\; [\,2\,b + k\,]\left[\,k_N\left(k_A + k^R\right) + k_A\, k^R\,\right] + b\,k_A\,[\,3\,b + 2\,k\,] - b^2\,[\,b + k\,] \;>\; 0\,. \tag{2}$$

$$k_A\,[\,(a - c_N)\,(2\,b + k) - b\,(a - c)\,] \;>\; [\,c_N - c_A\,]\left[\,3\,b^2 + 2\,b\left(k + k^R\right) + k\,k^R\,\right]. \tag{3}$$

R's problem is:

$$\underset{q_A \geq 0,\; q_N \geq 0}{\text{Maximize}} \quad P_A(\,q_A + q_N + q\,)\,q_A + [\,a - b\,(\,q_A + q_N + q\,)\,]\,q_N - C^R(q_A, q_N)$$

$$\text{where} \quad P_A(Q) \;=\; \begin{cases} \bar{p} & \text{if } P(Q) \,\geq\, \bar{p} \\ P(Q) & \text{if } P(Q) \,<\, \bar{p}\,. \end{cases} \tag{4}$$

The rival's problem is:

$$\underset{q \geq 0}{\text{Maximize}} \quad [\,a - b\,(\,q_A + q_N + q\,)\,]\,q \,-\, C(q)\,. \tag{5}$$

Proposition 1. *There exist values of the price cap, $0 < \overline{p}_1 < \overline{p}_2 < \overline{p}_3$, such that, in equilibrium, $q_A = 0$ if and only if $\overline{p} \leq \overline{p}_1$. Furthermore: (i) $\overline{p} < P(Q)$ if $\overline{p} \leq \overline{p}_2$; (ii) $\overline{p} = P(Q)$ if $\overline{p} \in (\overline{p}_2, \overline{p}_3\,]$; and (iii) $\overline{p} > P(Q)$ if $\overline{p} > \overline{p}_3$.*

Proof. The proof follows directly from Lemmas A1 – A6 (below), which refer to the following definitions.

$$\bar{p}_1 \;\equiv\; c_A + \frac{[\,a - c_N\,]\,[\,2\,b + k\,] \,-\, b\,[\,a - c\,]}{[\,2\,b + k_N + k^R\,]\,[\,2\,b + k\,] - b^2}\;\left[\,b + k^R\,\right]. \tag{6}$$

$$\begin{aligned} \bar{p}_2 \;\equiv\; \frac{1}{D_2}\,\Big\{\, & [\,a\,(b + k) + b\,c\,]\left[\,\left(b + k^R\right)(k_N + k_A) + k_N\,k_A - b\,k_N\,\right] \\ & +\; b\,[\,b + k\,]\,[\,k_A - b\,]\,c_N + b\,[\,k_N + b\,]\,[\,b + k\,]\,c_A\,\Big\} \end{aligned}$$

$$\begin{aligned} \text{where}\;\; D_2 \;\equiv\;& b\,[\,b + k\,]\,[\,k_N + k_A\,] + k_N\,[\,k_A - b\,]\,[\,2\,b + k\,] \\ & +\; [\,k_N + k_A\,]\,[\,2\,b + k\,]\left[\,b + k^R\,\right]. \end{aligned} \tag{7}$$

$$\begin{aligned} \bar{p}_3 \;\equiv\; \frac{1}{D_3}\,\Big\{\, & [\,a\,(b + k) + b\,c\,]\left[\,\left(b + k^R\right)(k_N + k_A) + k_N\,k_A\,\right] \\ & +\; b\,c_N\,[\,b + k\,]\,k_A + b\,k_N\,[\,b + k\,]\,c_A\,\Big\} \end{aligned}$$

where $D_3 \equiv b\,[\,b+k\,]\,[\,k_N+k_A\,]+k_N\,k_A\,[\,2\,b+k\,]$

$$+\,[\,k_N+k_A\,]\,[\,2\,b+k\,]\left[\,b+k^R\,\right] \;=\; D_2+b\,k_N\,[\,2\,b+k\,]\,. \qquad (8)$$

Lemma A1. Suppose $\bar{p} \leq \bar{p}_1$. Then in equilibrium:

$$q_A \;=\; 0\,, \quad q_N \;=\; \frac{[\,a-c_N\,]\,[\,2\,b+k\,]-b\,[\,a-c\,]}{[\,2\,b+k_N+k^R\,]\,[\,2\,b+k\,]-b^2}\,,$$

$$q \;=\; \frac{[\,a-c\,]\left[\,2\,b+k_N+k^R\,\right]-b\,[\,a-c_N\,]}{[\,2\,b+k_N+k^R\,]\,[\,2\,b+k\,]-b^2}\,, \text{ and}$$

$$Q \;=\; q_A+q_N+q \;=\; \frac{[\,a-c\,]\left[\,b+k_N+k^R\,\right]+[\,a-c_N\,]\,[\,b+k\,]}{[\,2\,b+k_N+k^R\,]\,[\,2\,b+k\,]-b^2}\,. \qquad (9)$$

Proof. (4) implies that R's problem when $q_A = 0$ is:

$$\underset{q_N \geq 0}{\text{Maximize}}\;[\,a-b\,(q_N+q)-c_N\,]\,q_N-\frac{k_N}{2}\,(q_N)^2-\frac{k^R}{2}\,(q_N)^2\,. \qquad (10)$$

(10) implies that R's profit-maximizing choice of $q_N > 0$ is determined by:

$$a-2\,b\,q_N-b\,q-c_N-k_N\,q_N-k^R\,q_N \;=\; 0 \;\;\Rightarrow\;\; q_N \;=\; \frac{a-c_N-b\,q}{2\,b+k_N+k^R}\,. \qquad (11)$$

(5) implies that the necessary condition for an interior solution to the rival's problem is:

$$a-b\,[\,q_A+q_N+q\,]-c-b\,q-k\,q \;=\; 0 \;\;\Leftrightarrow\;\; [\,2\,b+k\,]\,q \;=\; a-b\,[\,q_A+q_N\,]-c$$

$$\Leftrightarrow\;\; q \;=\; \frac{a-c}{2\,b+k}-\frac{b}{2\,b+k}\,[\,q_A+q_N\,]\,. \qquad (12)$$

(11) and (12) imply that when $q_A = 0$:

$$q_N \;=\; \frac{a-c_N}{2\,b+k_N+k^R}-\frac{b}{2\,b+k_N+k^R}\left[\frac{a-c-b\,q_N}{2\,b+k}\right]$$

$$=\; \frac{[\,a-c_N\,]\,[\,2\,b+k\,]-b\,[\,a-b\,q_N-c\,]}{[\,2\,b+k_N+k^R]\,[\,2\,b+k\,]}$$

$$\Rightarrow\;\; q_N\left[1-\frac{b^2}{[\,2\,b+k_N+k^R]\,[\,2\,b+k\,]}\right] \;=\; \frac{[\,a-c_N\,]\,[\,2\,b+k\,]\;-\;b\,[\,a-c\,]}{[\,2\,b+k_N+k^R\,]\,[\,2\,b+k\,]}$$

$$\Rightarrow\;\; q_N\left[\,(2\,b+\;k_N)\,(2\,b\;+\;k)-b^2\,\right] \;=\; [\,a-c_N\,]\,[\,2\,b+k\,]-b\,[\;a-c\,]$$

$$\Rightarrow\;\; q_N \;=\; \frac{[\,a-c_N\,]\,[\,2\,b+k\,]-b\,[\,a-c\,]}{[\,2\,b+k_N+k^R\,]\,[\,2\,b+k\,]-b^2}\,. \qquad (13)$$

(12) and (13) imply:

$$
\begin{aligned}
q \;&=\; \frac{a-c}{2\,b+k} - \left[\frac{b}{2\,b+k}\right] \frac{[\,a-c_N\,]\,[\,2\,b+k\,]-b\,[\,a-c\,]}{[\,2\,b+k_N+k^R\,]\,[\,2\,b+k\,]-b^2} \\
&=\; \frac{[\,a-c\,]\left[\,\left(2\,b+k_N+k^R\right)(2\,b+k)-b^2\,\right]-b\,[\,a-c_N\,]\,[\,2\,b+k\,]+b^2\,[\,a-c\,]}{[\,2\,b+k\,]\,[\,[\,2\,b+k_N+k^R\,]\,[\,2\,b+k\,]-b^2\,]} \\
&=\; \frac{[\,a-c\,]\left[\,2\,b+k_N+k^R\right][\,2\,b+k\,]-b\,[\,a-c_N\,]\,[\,2\,b+k\,]}{[\,2\,b\,+\,k\,]\,[\,[\,2\,b+k_N+k^R\,]\,[\,2\,b+k\,]\,-\,b^2\,]} \\
&=\; \frac{[\,a-c\,]\left[\,2\,b+k_N+k^R\right]-b\,[\,a-c_N\,]}{[\,2\,b+k_N+k^R\,]\,[\,2\,b+k\,]-b^2}\,. \qquad (14)
\end{aligned}
$$

(13) and (14) imply:

$$
Q \;=\; q+q_N \;=\; \frac{[\,a-c\,]\left[\,b+k_N+k^R\right]+[\,a-c_N\,]\,[\,b+k\,]}{[\,2\,b+k_N+k^R\,]\,[\,2\,b+k\,]-b^2}\,. \qquad (15)
$$

From (6):

$$
\begin{aligned}
\bar{p}_1 \;&=\; \frac{1}{[\,2\,b+k_N+k^R\,]\,[\,2\,b+k\,]-b^2} \\
&\quad\cdot \left\{\,[\,a-c_N\,]\,[\,2\,b+k\,]\left[\,b+k^R\right]-b\,[\,a-c\,]\left[\,b+k^R\right]\right. \\
&\qquad\qquad \left. +\; c_A\left[\,\left(2\,b+k_N^R+k^R\right)(2\,b\,+\,k)-b^2\,\right]\right\}. \qquad (16)
\end{aligned}
$$

(15) implies:

$$
\begin{aligned}
P(Q) \;&=\; a-b\,\frac{[\,a-c\,]\left[\,b+k_N+k^R\right]+[\,a-c_N\,]\,[\,b+k\,]}{[\,2\,b+k_N+k^R\,]\,[\,2\,b+k\,]-b^2} \\
&=\; \frac{a\left[\,\left(2\,b+k_N+k^R\right)(2\,b+k)-b^2\,\right]-b\,[\,a-c\,]\left[\,b+k_N+k^R\right]-b\,[\,a-c_N\,]\,[\,b+k\,]}{[\,2\,b+k_N+k^R\,]\,[\,2\,b+k\,]-b^2}\,. \qquad (17)
\end{aligned}
$$

Observe that: $\left[\,2\,b+k_N+k^R\right][\,2\,b+k\,] \;>\; 4\,b^2 \;>\; b^2\,.$

Therefore, (16) and (17) imply:

$$
\begin{aligned}
\bar{p}_1 \;<\; P(Q) \;\;\Leftrightarrow\;\; & [\,a-c_N\,]\,[\,2\,b+k\,]\left[\,b+k^R\right]-b\,[\,a-c\,]\left[\,b+k^R\right] \\
& \quad +\; c_A\left[\,\left(2\,b+k_N+k^R\right)(2\,b\,+\,k)-b^2\,\right] \\
& <\; a\left[\,\left(2\,b+k_N+k^R\right)(2\,b+k)-b^2\,\right] \\
& \qquad -\; b\,[\,a-c\,]\left[\,b+k_N+k^R\right]-b\,[\,a-c_N\,]\,[\,b+k\,] \\
\Leftrightarrow\;\; & [\,a-c_N\,]\,[\,2\,b+k\,]\left[\,b+k^R\right]-b\,[\,a-c\,]\left[\,b+k^R\right]
\end{aligned}
$$

$$+ \, c_A \left[\left(2\,b + k_N + k^R\right)(2\,b + k) - b^2 \right]$$

$$< \; a \left[\left(2\,b + k_N + k^R\right)(2\,b + k) - b^2 \right]$$

$$- \;\; b\,[\,a - c\,] \left[b + k_N + k^R \right] - b\,[\,a - c_N\,]\,[\,b + k\,]$$

$$\Leftrightarrow \;\; 0 \; < \; [\,a - c_A\,] \left[\left(2\,b + k_N \, + \, k^R\right)(2\,b + k) - b^2 \right] - b\,[\,a - c\,]\,k_N$$

$$- \; [\,a - c_N\,] \left[(2\,b + k)\left(b + k^R\right) + b\,(b + k) \right]$$

$$\Leftrightarrow \;\; 0 \; < \; [\,a \, - c_A\,] \left[2\,b\,k + 2\,b\,k^R + k\,k^R + 3\,b^2 + 2\,b\,k_N + k\,k_N \right]$$

$$- \;\; b\,[\,a - c\,]\,k_N - [\,a - c_N\,] \left[2\,b\,k + 2\,b\,k^R + k\,k^R + 3\,b^2 \right]$$

$$\Leftrightarrow \;\; [\,c_N - c_A\,] \left[2\,b\,k + 2\,b\,k^R + k\,k^R + 3\,b^2 \right]$$

$$+ \, k_N \left[(a - c_A)(2\,b + k) - b\,(a - c) \right] \; > \; 0\,. \tag{18}$$

The last inequality in (18) reflects (3). Therefore, $\bar{p} < P(Q)$ when $\bar{p} \leq \bar{p}_1$.

It remains to show that $q_A = 0$ when $\bar{p} \leq \bar{p}_1$. Because $\bar{p} < P(Q)$ when $\bar{p} \leq \bar{p}_1$, $q_A = 0$ when:

$$\frac{\partial}{\partial q_A} \Big\{ \; [\,\bar{p} - c_A\,]\,q_A + [\,a - b\,(q_A + q_N + q) - c_N\,]\,q_N$$

$$- \; \frac{k_A}{2}\,[\,q_A\,]^2 - \frac{k_N}{2}\,[\,q_N\,]^2 - \frac{k^R}{2}\,[\,q_N + q_A\,]^2 \Big\} \Big|_{q_A = 0} \;\; \leq \;\; 0$$

$$\Leftrightarrow \;\; \bar{p} - c_A - b\,q_N - k^R q_N \;\; \leq \;\; 0$$

$$\Leftrightarrow \;\; \bar{p} \;\; \leq \;\; c_A + \frac{[\,a \, - \, c_N\,]\,[\,2\,b + k\,] - b\,[\,a - c\,]}{[\,2\,b + k_N + k^R\,]\,[\,2\,b + k\,] - b^2} \left[b + k^R \right] \;\; = \;\; \bar{p}_1\,. \tag{19}$$

The equality in (19) reflects (13). $\square$

Lemma A2. *Suppose* $\bar{p} \in (\,\bar{p}_1, \bar{p}_2\,]$. *Then in equilibrium*:

$$q_A \;\; = \;\; \frac{1}{D} \left\{ \left[3\,b^2 + 2\,b\left(k + k_N + k^R\right) + k\left(k_N + k^R\right) \right] [\,\bar{p} - c_A\,] \right.$$

$$\left. + \;\; b\left[b + k^R \right] [\,a - c\,] - [\,2\,b + k\,] \left[b + k^R \right] [\,a - c_N\,] \right\}; \tag{20}$$

$$q_N \;\; = \;\; \frac{1}{D} \; \left\{ [\,2\,b + k\,] \left[k_A + k^R \right] [\,a - c_N\,] - b \left[k_A + k^R \right] [\,a - c\,] \right.$$

$$\left. - \;\; \left[b\left(b + 2\,k^R\right) + k\left(b + k^R\right) \right] [\,\bar{p} - c_A\,] \right\}; \tag{21}$$

$$Q^R \;\; \equiv \;\; q_A + q_N \;\; = \;\; \frac{1}{D} \left\{ [\,2\,b + k\,]\,[\,b + k_N\,]\,[\,\bar{p} - c_A\,] + [\,2\,b + k\,]\,[\,k_A - b\,]\,[\,a - c_N\,] \right.$$

$$- \; b\,[\,k_A - b\,]\,[\,a - c\,]\,\}\,; \tag{22}$$

$$q \;=\; \frac{1}{D}\,\{\,\big[\,k_N\left(k_A + k^R\right) + k_A\,k^R + 2\,b\,k_A - b^2\,\big]\,[\,a - c\,]$$
$$- \; b\,[\,k_A - b\,]\,[\,a - c_N\,] - b\,[\,b + k_N\,]\,[\,\overline{p} - c_A\,]\,\}\,; \text{ and} \tag{23}$$

$$Q \;=\; q + q_A + q_N \;=\; \frac{1}{D}\,\{\,[\,b + k\,]\,[\,b + k_N\,]\,[\,\overline{p} - c_A\,] + [\,b + k\,]\,[\,k_A - b\,]\,[\,a - c_N\,]$$
$$+ \;\big[\,k^R\,(\,k_A + k_N\,) + k_A\,(\,b + k_N\,)\,\big]\,[\,a - c\,]\,\}\,. \tag{24}$$

<u>Proof</u>. (4) implies that if $q_A > 0$ and $\overline{p} < P(Q)$, R's problem, [P-R], is:

$$\underset{q_A,\,q_N}{\text{Maximize}} \;\; \overline{p}\,q_A + [\,a - b\,(\,q_A + q_N + q\,)\,]\,q_N - c_A\,q_A - \frac{k_A}{2}\,[\,q_A\,]^2$$
$$- \; c_N\,q_N - \frac{k_N}{2}\,[\,q_N\,]^2 - \frac{k^R}{2}\,[\,q_A + q_N\,]^2\,.$$

The necessary conditions for a solution to [P-R] in this case are:[1]

$$q_A: \qquad \overline{p} - b\,q_N - c_A - k_A\,q_A - k^R\,[\,q_A + q_N\,] \;=\; 0\,; \tag{25}$$

$$q_N: \qquad a - b\,[\,q_A + q_N + q\,] - b\,q_N - c_N - k_N\,q_N - k^R\,[\,q_A + q_N\,] \;=\; 0\,. \tag{26}$$

(25) implies:

$$\overline{p} - b\,q_N - c_A - k^R\,q_N \;=\; \big[\,k_A + k^R\,\big]\,q_A \;\;\Rightarrow\;\; q_A \;=\; \frac{\overline{p} - c_A}{k_A + k^R} - \left[\frac{b + k^R}{k_A + k^R}\right] q_N\,. \tag{27}$$

(26) implies:

$$a - b\,[\,q_A + q\,] - c_N - k^R\,q_A \;=\; \big[\,2\,b + k_N + k^R\,\big]\,q_N$$
$$\Rightarrow\;\; q_N \;=\; \frac{a - c_N}{2\,b + k_N + k^R} - \frac{\big[\,b + k^R\,\big]\,q_A + b\,q}{2\,b + k_N + k^R}\,. \tag{28}$$

(25) also implies:

$$\overline{p} - c_A - k_A\,q_A - k^R\,q_A \;=\; \big[\,b + k^R\,\big]\,q_N \;\;\Rightarrow\;\; q_N \;=\; \frac{\overline{p} - c_A}{b + k^R} - \left[\frac{k_A + k^R}{b + k^R}\right] q_A\,. \tag{29}$$

(28) and (29) imply:

$$\frac{a - c_N}{2\,b + k_N + k^R} - \frac{\big[\,b + k^R\,\big]\,q_A + b\,q}{2\,b + k_N + k^R} \;=\; \frac{\overline{p} - c_A}{b + k^R} - \left[\frac{k_A + k^R}{b + k^R}\right] q_A$$

$$\Rightarrow\;\; \left[\frac{b + k^R}{2\,b + k_N + k^R} - \frac{k_A + k^R}{b + k^R}\right] q_A \;=\; \frac{a - c_N}{2\,b + k_N + k^R} - \frac{\overline{p} - c_A}{b + k^R} - \frac{b\,q}{2\,b + k_N + k^R}$$

[1] It is readily verified that the determinant of the Hessian associated with [P-R] in this setting is $\big[\,k_A + k^R\,\big]\,\big[\,2\,b + k_N + k^R\,\big] - \big[\,b + k^R\,\big]^2$, which is strictly positive if $k_A \,\geq\, \frac{b}{2}$.

$$\Rightarrow \quad \Big\{ \big[b + k^R \big]^2 - \big[k_A + k^R \big] \big[2b + k_N + k^R \big] \Big\} q_A$$

$$= \big[b + k^R \big] [a - c_N] - \big[2b + k_N + k^R \big] [\overline{p} - c_A] - b \big[b + k^R \big] q$$

$$\Rightarrow \quad q_A = \frac{\big[b + k^R \big] [a - c_N] - \big[2b + k_N + k^R \big] [\overline{p} - c_A] - b \big[b + k^R \big] q}{[b + k^R]^2 - [k_A + k^R] [2b + k_N + k^R]} . \tag{30}$$

(5) implies that the rival's problem in this setting, [P], is:

$$\underset{q}{\text{Maximize}} \quad [a - b (q_A + q_N + q) - c] q - \frac{k}{2} (q)^2 . \tag{31}$$

The necessary condition for an interior solution to [P] is:

$$a - b [q_A + q_N + q] - c - b q - k q = 0 \quad \Leftrightarrow \quad [2b + k] q = a - b [q_A + q_N] - c$$

$$\Leftrightarrow \quad q = \frac{a - c}{2b + k} - \frac{b}{2b + k} [q_A + q_N] . \tag{32}$$

(29) and (32) imply:

$$q = \frac{a - c}{2b + k} - \frac{b}{2b + k} \left[q_A + \frac{\overline{p} - c_A}{b + k^R} - \left(\frac{k_A + k^R}{b + k^R} \right) q_A \right]$$

$$= \frac{a - c}{2b + k} - \frac{b}{2b + k} \left[\frac{\overline{p} - c_A}{b + k^R} \right] - \frac{b}{2b + k} \left[1 - \frac{k_A + k^R}{b + k^R} \right] q_A$$

$$= \frac{a - c}{2b + k} - \frac{b}{2b + k} \left[\frac{\overline{p} - c_A}{b + k^R} \right] - \frac{b}{2b + k} \left[\frac{b - k_A}{b + k^R} \right] q_A . \tag{33}$$

(30) and (33) imply:

$$q_A = \frac{\big[b + k^R \big] [a - c_N] - \big[2b + k_N + k^R \big] [\overline{p} - c_A]}{[b + k^R]^2 - [k_A + k^R] [2b + k_N + k^R]}$$

$$- \frac{b \big[b + k^R \big]}{[b + k^R]^2 - [k_A + k^R] [2b + k_N + k^R]}$$

$$\cdot \left\{ \frac{a - c}{2b + k} - \frac{b}{2b + k} \left[\frac{\overline{p} - c_A}{b + k^R} \right] - \frac{b}{2b + k} \left[\frac{b - k_A}{b + k^R} \right] q_A \right\}$$

$$\Rightarrow \quad q_A \left[1 - \left(\frac{b \big[b + k^R \big]}{[b + k^R]^2 - [k_A + k^R] [2b + k_N + k^R]} \right) \left(\frac{b}{2b + k} \right) \left(\frac{b - k_A}{b + k^R} \right) \right]$$

$$= \frac{\big[b + k^R \big] [a - c_N] - \big[2b + k_N + k^R \big] [\overline{p} - c_A]}{[b + k^R]^2 - [k_A + k^R] [2b + k_N + k^R]}$$

$$- \frac{b\left[\, b + k^R \,\right]}{[\, b + k^R\,]^2 - [\, k_A + k^R\,]\,[\, 2\,b + k_N + k^R\,]} \left[\frac{a - c}{2\,b + k} - \frac{b}{2\,b + k} \left(\frac{\overline{p} - c_A}{b + k^R} \right) \right]$$

$$\Rightarrow \quad q_A \left[1 - \frac{b^2\,[\, b - k_A\,]}{[\, 2\,b + k\,]\left\{[\, b + k^R\,]^2 - [\, k_A + k^R\,]\,[\, 2\,b + k_N + k^R\,]\right\}} \right]$$

$$= \frac{[\, 2\,b + k\,]\left\{ \left[\, b + k^R \,\right][\, a - c_N\,] - \left[\, 2\,b + k_N + k^R \,\right][\, \overline{p} - c_A\,] \right\}}{[\, 2\,b + k\,]\left\{[\, b + k^R\,]^2 - [\, k_A + k^R\,]\,[\, 2\,b + k_N + k^R\,]\right\}}$$

$$- \frac{b\left[\, b + k^R \,\right]\left[\, a - c - b\left(\frac{\overline{p} - c_A}{b + k^R}\right)\right]}{[\, 2\,b + k\,]\left\{[\, b + k^R\,]^2 - [\, k_A + k^R\,]\,[\, 2\,b + k_N + k^R\,]\right\}}$$

$$\Rightarrow \quad q_A \left\{ [\, 2\,b + k\,] \left(\left[\, b + k^R \,\right]^2 - \left[\, k_A + k^R \,\right] \left[\, 2\,b + k_N + k^R \,\right] \right) - b^2\,[\, b - k_A\,] \right\}$$

$$= [\, 2\,b + k\,] \left\{ \left[\, b + k^R \,\right][\, a - c_N\,] - \left[\, 2\,b + k_N + k^R \,\right][\, \overline{p} - c_A\,] \right\}$$

$$- b\left[\, (\, a - c\,)\left(b + k^R\right) - b\,(\, \overline{p} - c_A\,) \right]. \tag{34}$$

Observe that:

$$[\, 2\,b + k\,] \left\{ \left[\, b + k^R \,\right]^2 - \left[\, k_A + k^R \,\right] \left[\, 2\,b + k_N + k^R \,\right] \right\} - b^2\,[\, b - k_A\,]$$

$$= [\, 2\,b + k\,] \left\{ b^2 + 2\,b\,k^R + \left(k^R\right)^2 - 2\,b\,k_A - 2\,b\,k^R - k_A\,k_N - k^R k_N - k_A\,k^R - \left(k^R\right)^2 \right\}$$

$$- b^3 + b^2 k_A$$

$$= [\, 2\,b + k\,] \left[b^2 - 2\,b\,k_A - k_A\,k_N - k^R k_N - k_A\,k^R \right] - b^3 + b^2 k_A$$

$$= 2\,b^3 - 4\,b^2 k_A - 2\,b\,k_A\,k_N - 2\,b\,k^R k_N - 2\,b\,k_A\,k^R$$

$$+ b^2 k - 2\,b\,k\,k_A - k\,k_A\,k_N - k\,k^R k_N - k\,k_A\,k^R - b^3 + b^2 k_A$$

$$= b^3 - 3\,b^2 k_A - 2\,b\,k_A\,k_N - 2\,b\,k^R k_N - 2\,b\,k_A\,k^R$$

$$+ b^2 k - 2\,b\,k\,k_A - k\,k_A\,k_N - k\,k^R k_N - k\,k_A\,k^R$$

$$= b^2\,[\, b + k\,] - b\,k_A\,[\, 3\,b + 2\,k\,] - [\, 2\,b + k\,] \left[\, k_N \left(k_A + k^R\right) + k_A\,k^R \,\right]. \tag{35}$$

Further observe that:

$$[\, 2\,b + k\,] \left[\, 2\,b + k_N + k^R \,\right] - b^2 = 2\,b \left[\, 2\,b + k_N + k^R \,\right] + k \left[\, 2\,b + k_N + k^R \,\right] - b^2$$

$$= 3\,b^2 + 2\,b \left[\, k_N + k^R \,\right] + k \left[\, 2\,b + k_N + k^R \,\right]$$

$$= 3\,b^2 + 2\,b \left[\, k + k_N + k^R \,\right] + k \left[\, k_N + k^R \,\right]. \tag{36}$$

(2) and (34) – (36) imply:

$$q_A = \frac{1}{D} \{ \left[3\,b^2 + 2\,b \left(k + k_N + k^R \right) + k \left(k_N + k^R \right) \right] [\,\overline{p} - c_A\,]$$

$$+ \; b \left[b + k^R \right] [\,a - c\,] - [\,2\,b + k\,] \left[b + k^R \right] [\,a - c_N\,] \}. \tag{37}$$

(2), (29), and (37) imply:

$$q_N = \frac{\overline{p} - c_A}{b + k^R} - \left[\frac{k_A + k^R}{b + k^R} \right] \frac{1}{D} \{ \left[3\,b^2 + 2\,b \left(k + k_N + k^R \right) + k \left(k_N + k^R \right) \right] [\,\overline{p} - c_A\,]$$

$$+ \; b \left[b + k^R \right] [\,a - c\,] - [\,2\,b + k\,] \left[b + k^R \right] [\,a - c_N\,] \}$$

$$= \frac{1}{D\,[\,b + k^R\,]} \{ [\,\overline{p} - c_A\,]\,D$$

$$- \; \left[k_A + k^R \right] \left[3\,b^2 + 2\,b \left(k + k_N + k^R \right) + k \left(k_N + k^R \right) \right] [\,\overline{p} - c_A\,]$$

$$- \; b \left[b + k^R \right] \left[k_A + k^R \right] [\,a - c\,]$$

$$+ \; [\,2\,b + k\,] \left[k_A + k^R \right] \left[b + k^R \right] [\,a - c_N\,] \}. \tag{38}$$

(2) implies:

$$D - \left[k_A + k^R \right] \left[3\,b^2 + 2\,b \left(k + k_N + k^R \right) + k \left(k_N + k^R \right) \right]$$

$$= [\,2\,b + k\,] \left[k_N \left(k_A + k^R \right) + k_A\,k^R \right] + b\,k_A\,[\,3\,b + 2\,k\,] - b^2\,[\,b + k\,]$$

$$- \; \left[k_A + k^R \right] \left[3\,b^2 + 2\,b \left(k + k_N + k^R \right) + k \left(k_N + k^R \right) \right]$$

$$= \left[k_A + k^R \right] \left[(\,2\,b + k\,)\,k_N - 3\,b^2 - 2\,b \left(k + k_N + k^R \right) - k \left(k_N + k^R \right) \right]$$

$$+ \; [\,2\,b + k\,]\,k_A\,k^R + 3\,b^2 k_A + 2\,b\,k\,k_A - b^3 - b^2 k$$

$$= \left[k_A + k^R \right] \left[-\,3\,b^2 - 2\,b\,k - 2\,b\,k^R - k\,k^R \right]$$

$$+ \; 2\,b\,k_A\,k^R + k\,k_A\,k^R + 3\,b^2 k_A + 2\,b\,k\,k_A - b^3 - b^2 k$$

$$= -\,3\,b^2 k_A - 3\,b^2 k^R - 2\,b\,k\,k_A - 2\,b\,k\,k^R - \; 2\,b\,k_A\,k^R - 2\,b \left(k^R \right)^2 - k\,k_A\,k^R$$

$$- \; k \left(k^R \right)^2 + 2\,b\,k_A\,k^R + k\,k_A\,k^R + 3\,b^2 k_A + 2\,b\,k\,k_A - b^3 - b^2 k$$

$$= -\,3\,b^2 k^R - 2\,b\,k\,k^R - \; 2\,b\,k_A\,k^R - 2\,b \left(k^R \right)^2 - k \left(k^R \right)^2$$

$$+ \; 2\,b\,k_A\,k^R - b^3 - b^2 k$$

$$= -\,b^2 k^R - 2\,b^2 k^R - b\,k\,k^R - b\,k\,k^R - 2\,b \left(k^R \right)^2 - k \left(k^R \right)^2 - b^2 b - b\,k\,b$$

$$= \; - b^2 \left[b + k^R \right] - 2 b^2 k^R - b k \left[b + k^R \right] - b k k^R - 2 b \left(k^R \right)^2 - k \left(k^R \right)^2$$

$$= \; - b^2 \left[b + k^R \right] - 2 b k^R \left[b + k^R \right] - b k \left[b + k^R \right] - k k^R \left[b + k^R \right]$$

$$= \; - \left[b + k^R \right] \left[b^2 + 2 b k^R + b k + k k^R \right]$$

$$= \; - \left[b + k^R \right] \left[b \left(b + 2 k^R \right) + k \left(b + k^R \right) \right] . \tag{39}$$

(38) and (39) imply:

$$q_N \; = \; \frac{1}{D} \left\{ \left[2 b + k \right] \left[k_A + k^R \right] \left[a - c_N \right] - b \left[k_A + k^R \right] \left[a - c \right] \right.$$
$$\left. - \left[b \left(b + 2 k^R \right) + k \left(b + k^R \right) \right] \left[\overline{p} - c_A \right] \right\} . \tag{40}$$

Observe that:

$$3 b^2 + 2 b \left[k + k_N + k^R \right] + k \left[k_N + k^R \right] - \left[b \left(b + 2 k^R \right) + k \left(b + k^R \right) \right]$$

$$= \; 3 b^2 + 2 b k + 2 b k_N + 2 b k^R + k k_N + k k^R - b^2 - 2 b k^R - b k - k k^R$$

$$= \; 2 b^2 + b k + 2 b k_N + k k_N \; = \; b \left[2 b + k \right] + k_N \left[2 b + k \right] \; = \; \left[2 b + k \right] \left[b + k_N \right] . \tag{41}$$

Further observe that:

$$b \left[b + k^R \right] - b \left[k_A + k^R \right] \; = \; b \left[b - k_A \right] \quad \text{and}$$

$$\left[2 b + k \right] \left[k_A + k^R \right] - \left[2 b + k \right] \left[b + k^R \right] \; = \; \left[2 b + k \right] \left[k_A - b \right] . \tag{42}$$

(37) and (40) – (42) imply:

$$q_A + q_N \; = \; \frac{1}{D} \left\{ \left[2 b + k \right] \left[b + k_N \right] \left[\overline{p} - c_A \right] - b \left[k_A - b \right] \left[a - c \right] \right.$$
$$\left. + \left[2 b + k \right] \left[k_A - b \right] \left[a - c_N \right] \right\} . \tag{43}$$

(32) and (43) imply:

$$q \; = \; \frac{a - c}{2 b + k} - \left[\frac{b}{2 b + k} \right] \frac{1}{D} \left\{ \left[2 b + k \right] \left[b + k_N \right] \left[\overline{p} - c_A \right] - b \left[k_A - b \right] \left[a - c \right] \right.$$
$$\left. + \left[2 b + k \right] \left[k_A - b \right] \left[a - c_N \right] \right\}$$

$$= \; \frac{D + b^2 \left[k_A - b \right]}{D \left[2 b + k \right]} \left[a - c \right]$$
$$- \frac{b}{D} \left\{ \left[b + k_N \right] \left[\overline{p} - c_A \right] + \left[2 b + k \right] \left[k_A - b \right] \left[a - c_N \right] \right\} . \tag{44}$$

(2) implies:

$$D + b^2 \left[k_A - b \right] \; = \; \left[2 b + k \right] \left[k_N \left(k_A + k^R \right) + k_A k^R \right] + b k_A \left[3 b + 2 k \right]$$

$$
\begin{aligned}
& - b^2 [\, b + k \,] + b^2 [\, k_A - b \,] \\
= \; & [\, 2\,b + k \,] \left[\, k_N \left(k_A + k^R \right) + k_A\, k^R \,\right] + 4\, b^2 k_A + 2\, b\, k\, k_A - b^2 [\, 2\,b + k \,] \\
= \; & [\, 2\,b + k \,] \left[\, k_N \left(k_A + k^R \right) + k_A\, k^R \,\right] + 2\, b\, k_A [\, 2\,b + k \,] - b^2 [\, 2\,b + k \,] \\
= \; & [\, 2\,b + k \,] \left[\, k_N \left(k_A + k^R \right) + k_A\, k^R + 2\, b\, k_A - b^2 \,\right]. \qquad (45)
\end{aligned}
$$

(44) and (45) imply:

$$
\begin{aligned}
q \; = \; \frac{1}{D} \, \{ & \left[\, k_N \left(k_A + k^R \right) + k_A\, k^R + 2\, b\, k_A - b^2 \,\right] [\, a - c \,] \\
& - \; b\, [\, k_A - b \,] \, [\, a - c_N \,] - b\, [\, b + k_N \,] \, [\, \overline{p} - c_A \,] \, \} \,. \qquad (46)
\end{aligned}
$$

Observe that:

$$
\begin{aligned}
& [\, 2\,b + k \,] \, [\, b + k_N \,] - b\, [\, b + k_N \,] \; = \; [\, b + k \,] \, [\, b + k_N \,] \, ; \\
& [\, 2\,b + k \,] \, [\, k_A - b \,] - b\, [\, k_A - b \,] \; = \; [\, b + k \,] \, [\, k_A - b \,] \, ; \quad \text{and} \\
& k_N \left[\, k_A + k^R \,\right] + k_A\, k^R + 2\, b\, k_A - b^2 - b\, [\, k_A - b \,] \\
& \quad = \; k_N \left[\, k_A + k^R \,\right] + k_A\, k^R + b\, k_A \; = \; k^R\, [\, k_A + k_N \,] + k_A\, [\, b + k_N \,] \,. \qquad (47)
\end{aligned}
$$

(43), (46), and (47) imply:

$$
\begin{aligned}
Q \; = \; q + q_A + q_N \; = \; \frac{1}{D} \, \{ & [\, b + k \,] \, [\, b + k_N \,] \, [\, \overline{p} - c_A \,] + [\, b + k \,] \, [\, k_A - b \,] \, [\, a - c_N \,] \\
& + \left[\, k^R \, (\, k_A + k_N \,) + k_A \, (\, b + k_N \,) \,\right] [\, a - c \,] \, \} \,. \qquad (48)
\end{aligned}
$$

It remains to show that $q_A > 0$ and $\bar{p} \leq P(Q)$ when $\bar{p} \in (\, \bar{p}_1, \bar{p}_2 \,]$. (37) implies that $q_A > 0$ if:

$$
\begin{aligned}
& b\, [\, b + k_N \,] \, [\, a - c \,] + [\, \bar{p} - c_A \,] \left[\, 2\, b\, k + 2\, b\, k_N + 2\, b\, k^R + k\, k_N + k\, k^R + 3\, b^2 \,\right] \\
& \qquad - [\, a - c_N \,] \left[\, b\, k + 2\, b\, k^R + k\, k^R + 2\, b^2 \,\right] \; > \; 0 \\
\Leftrightarrow \quad & c_A + \frac{[\, a - c_N \,] \left[\, b\, k + 2\, b\, k^R + k\, k^R + 2\, b^2 \,\right] - b\, [\, b + k_N \,] \, [\, a - c \,]}{2\, b\, k + 2\, b\, k_N + 2\, b\, k^R + k\, k_N + k\, k^R + 3\, b^2} \; < \; \bar{p} \\
\Leftrightarrow \quad & \bar{p} \; > \; c_A + \frac{[\, a - c_N \,] \, [\, 2\, b + k \,] - b\, [\, a - c \,]}{[\, 2\, b + k_N + k^R \,] \, [\, 2\, b + k \,] - b^2} \left[\, b + k^R \,\right] \; = \; \bar{p}_1 \,.
\end{aligned}
$$

The equality here reflects (6). (48) implies:

$$
Q \; = \; \frac{1}{D} \, [\, C_1\, a + C_2\, c + C_3\, c_N + C_4\, c_A - C_4\, \bar{p} \,] \qquad (49)
$$

where $C_1 \; \equiv \; [\, b + k \,] \, [\, k_A - b \,] + k^R\, [\, k_A + k_N \,] + k_A\, [\, b + k_N \,]$

$$= b\,k_A + b\,k_A + k\,k_A - b\,k + k_A\,k^R + k_N\,k^R + b\,k_A + k_A\,k_N$$

$$= 2\,b\,k_A + k\,k_A + k_A\,k_N + k_A\,k^R + k_N\,k^R - b^2 - b\,k\,;$$

$$C_2 \;\equiv\; -k^R\,[\,k_A + k_N\,] - k_A\,[\,b + k_N\,]\,;\;\; C_3 \;\equiv\; -\,[\,b + k\,]\,[\,k_A - b\,]\,;\;\; \text{and}$$

$$C_4 \;\equiv\; -\,[\,b + k\,]\,[\,b + k_N\,]\,. \tag{50}$$

(49) implies:

$$P(Q) \;=\; a - b\,Q \;=\; \frac{[\,D - b\,C_1\,]\,a - b\,c\,C_2 - b\,C_3\,c_N - b\,C_4\,c_A + b\,C_4\;\bar{p}}{D}\,. \tag{51}$$

(2) and (50) imply:

$$D - b\,C_1 \;=\; [\,2\,b + k\,]\left[\,k_N\left(k_A + k^R\right) + k_A\,k^R\,\right] + b\,k_A\,[\,3\,b + 2\,k\,] - b^2\,[\,b + k\,]$$

$$-\;\; b\left[\,2\,b\,k_A + k\,k_A + k_A\,k_N + k_A\,k^R + k_N\,k^R - b^2 - b\,k\;\right]$$

$$=\; 2\,b\,k_A\,k_N + 2\,b\,k_N\,k^R + 2\,b\,k_A\,k^R + k\,k_A\,k_N + k\,k_N\,k^R + k\,k_A\,k^R + 3\,b^2 k_A$$

$$+\;\; 2\,b\,k\,k_A - b\,k - 2\,b^2 k_A - b\,k\,k_A - b\,k_A\,k_N - b\,k_A\,k^R - b\,k_N\,k^R + b^2 k$$

$$=\; b^2 k_A + b\,k\,k_A + b\,k_A\,k_N + b\,k_A\,k^R + b\,k_N\,k^R + k\,k_A\,k_N + k\,k_A\,k^R + k\,k_N\,k^R$$

$$=\; [\,b + k\,]\left[\,b\,k_A + k_A\,k_N + k_A\,k^R + k_N\,k^R\,\right]$$

$$=\; [\,b + k\,]\,[\,(b + k_A)\,(k_N + k_A) + k_N\,k_A - b\,k_N\,]\,. \tag{52}$$

(2) and (50) imply:

$$D \;-\; b\,C_4 \;=\; [\,2\,b + k\,]\left[\,k_N\left(k_A + k^R\right) + k_A\,k^R\,\right] + b\,k_A\,[\,3\,b + 2\,k\,] - b^2\,[\,b + k\,]$$

$$-\; b\,[\,b + k\,]\,[\,b + k_N\,]$$

$$=\; 3\,b^2 k_A - b^2 k - b^3 + 2\,b\,k\,k_A + 2\,b\,k_A\,k_N + 2\,b\,k_A\,k^R + 2\,b\,k_N\,k^R + k\,k_A\,k_N$$

$$+\;\; k\,k_A\,k^R + k\,k_N\,k^R + k_N\,b^2 + k_N\,k\,b + b^3 + b^2 k$$

$$=\; 3\,b^2 k_A + 2\,b\,k\,k_A + 2\,b\,k_A\,k_N + 2\,b\,k_A\,k^R + 2\,b\,k_N\,k^R + k\,k_A\,k_N$$

$$+\; k\,k_A\,k^R + k\,k_N\,k^R + k_N\,b^2 + k_N\,k\,b$$

$$=\; b\,[\,b + k\,]\,[\,k_N + k_A\,] + [\,k_A\,k_N - k_N\,b\,]\,[\,2\,b + k\,]$$

$$+\; [\,k_N + k_A\,]\,[\,2\,b + k\,]\left[\,b + k^R\,\right]. \tag{53}$$

(51) implies:

$$\bar{p} \;\leq\; P(Q) \;=\; \frac{[\,D - b\,C_1\,]\,a \;- b\,c\,C_2 -\; b\,C_3\,c_N \;-\; b\,C_4\,c_A + b\,C_4\;\bar{p}}{D}$$

$$\Leftrightarrow \quad \bar{p} - \frac{b\,C_4}{D}\,\bar{p} \;\leq\; \frac{[\,D - b\,C_1\,]\,a - b\,c\,C_2 - b\,C_3\,c_N - b\,C_4\,c_A}{D}$$

$$\Leftrightarrow \quad \bar{p}\left[1 - \frac{b\,C_4}{D}\right] \;\leq\; \frac{[\,D - b\,C_1\,]\,a - b\,c\,C_2 - b\,C_3\,c_N - b\,C_4\,c_A}{D} \tag{54}$$

$$\Leftrightarrow \quad \bar{p}\,[\,D - b\,C_4\,] \;\leq\; [\,D - b\,C_1\,]\,a - b\,c\,C_2 - b\,C_3\,c_N - b\,C_4\,c_A$$

$$\Leftrightarrow \quad \bar{p} \;\leq\; \frac{[\,D - b\,C_1\,]\,a - b\,c\,C_2 - b\,C_3\,c_N - b\,C_4\,c_A}{D - b\,C_4}\,. \tag{55}$$

(54) reflects the fact that $D - b\,C_4 > 0$ because $C_4 < 0$ (from (50)), and because $D > 0$, by assumption.

(50), (52), (53), and (55) imply:

$$\bar{p} \;\leq\; \frac{1}{D_2}\,\{\,a\,[\,b + k\,]\,\big[\,\big(b + k^R\big)\,(k_N + k_A) + k_N\,k_A - b\,k_N\,\big]$$
$$+\; b\,c\,\big[\,(k_N + k_A)\,\big(b + k^R\big) + k_A\,k_N - b\,k_N\,\big]$$
$$+\; b\,[\,b + k\,]\,[\,k_A - b\,]\,c_N + b\,[\,k_N + b\,]\,[\,b + k\,]\,c_A\,\}$$

$$\Leftrightarrow \quad \bar{p} \;\leq\; \frac{1}{D_2}\,\{\,[\,(b + k)\,a + b\,c\,]\,\big[\,\big(b + k^R\big)\,(k_N + k_A) + k_N\,k_A - b\,k_N\,\big]$$
$$+\; b\,[\,b + k\,]\,[\,k_A - b\,]\,c_N + b\,[\,k_N + b\,]\,[\,b + k\,]\,c_A\,\} \;=\; \bar{p}_2\,. \tag{56}$$

The equality in (56) reflects (7). (55) and (56) imply that $\bar{p} \leq P(Q)$ (and $q_A > 0$) when $\bar{p} \in (\,\bar{p}_1, \bar{p}_2\,]$. □

Lemma A3. *Suppose $\bar{p} \in (\,\bar{p}_2, \bar{p}_3\,]$, where $\bar{p}_2 < \bar{p}_3$. Then in equilibrium, $P(Q) = \bar{p}$. Furthermore*:

$$q_A \;=\; \frac{b\,[\,b + k\,]\,[\,c_N - c_A\,] + k_N\,[\,a - \bar{p}\,]\,[\,b + k\,] - b\,k_N\,[\,\bar{p} - c\,]}{b\,[\,b + k\,]\,[\,k_N + k_A\,]}\,;$$

$$q_N \;=\; \frac{k_A\,[\,b + k\,]\,[\,a - \bar{p}\,] - b\,k_A\,[\,\bar{p} - c\,] - b\,[\,b + k\,]\,[\,c_N - c_A\,]}{b\,[\,b + k\,]\,[\,k_N + k_A\,]}\,;$$

$$Q^R \;\equiv\; q_A + q_N \;=\; \frac{[\,b + k\,]\,[\,a - \bar{p}\,] - b\,[\,\bar{p} - c\,]}{b\,[\,b + k\,]}\,;$$

$$q \;=\; \frac{\bar{p} - c}{b + k}\,; \quad \text{and} \quad Q \;\equiv\; \frac{a - \bar{p}}{b}\,. \tag{57}$$

Proof. (4) implies that R's problem, [P-R], can be written as:

$$\underset{q_A,\;Q^R}{\text{Maximize}} \;\; \Pi_R \;\equiv\; \big[\,P_A(q + Q^R) - c_A\,\big]\,q_A + \big[\,P(Q^R + q) - c_N\,\big]\,\big[\,Q^R - q_A\,\big]$$

$$- \; \frac{k_A}{2}\left[\,q_A\,\right]^2 - \frac{k_N}{2}\left[\,Q^R - q_A\,\right]^2 - \frac{k^R}{2}\left[\,Q^R\,\right]^2$$

$$\text{where} \;\; P_A(q \,+\, Q^R) \;=\; \begin{cases} \bar{p} & \text{if } P(\,q + Q^R\,) \,\geq\, \bar{p} \\ P(\,q + Q^R\,) & \text{if } \bar{p} \,>\, P(\,q + Q^R\,). \end{cases} \tag{58}$$

(58) implies that the necessary conditions for a solution to [P-R] are:

$$\frac{\partial \Pi_R}{\partial q_A} \;=\; P_A\left(q + Q^R\right) - c_A - k_A\,q_A - \left[\,P\left(q + Q^R\right) - c_N\,\right] + k_N\left[\,Q^R - q_A\,\right] \;=\; 0 \tag{59}$$

$$\text{and} \quad \frac{\partial^+ \Pi_R}{\partial Q^R} \;\leq\; 0 \;<\; \frac{\partial^- \Pi_R}{\partial Q^R}\,, \tag{60}$$

where $\frac{\partial^- \Pi_R}{\partial Q^R}$ denotes the left-sided derivative of Π_R with respect to Q^R, which is relevant when $P_A(\cdot) \,=\, \bar{p}$, and $\frac{\partial^+ \Pi_R}{\partial Q^R}$ denotes the right-sided derivative of Π_R with respect to Q^R, which is relevant when $P_A(\cdot) = P(Q)$.

(12) implies:

$$a - b\,Q \;-b\,q - c - k\,q \;=\; 0$$

$$\Leftrightarrow \;\; \bar{p} - b\,q - c - k\,q \;=\; 0 \quad \Leftrightarrow \quad q \;=\; \frac{\bar{p} - c}{b + k}\,. \tag{61}$$

Because $\bar{p} \,=\, a - b\left[\,q + Q^R\,\right]$, (61) implies:

$$\bar{p} \;=\; a - b\left[\frac{\bar{p} - c}{b + k} + Q^R\right] \quad \Leftrightarrow \;\; b\,Q^R \;=\; a - \bar{p} - b\left[\frac{\bar{p} - c}{b + k}\right]$$

$$\Leftrightarrow \;\; Q^R \;=\; \frac{a - \bar{p}}{b} - \frac{\bar{p} - c}{b + k} \;=\; \frac{[\,a - \bar{p}\,]\,[\,b + k\,] - b\,[\,\bar{p} - c\,]}{b\,[\,b + k\,]}\,. \tag{62}$$

Because $\bar{p} \,=\, P_A(q + Q^R)$ in equilibrium, by assumption, (59) holds if:

$$\bar{p} - c_A - k_A\,q_A - [\,\bar{p} - c_N\,] + k_N\left[\,Q^R - q_A\,\right] \;=\; 0$$

$$\Leftrightarrow \;\; c_N - c_A - k_A\,q_A + k_N\,Q^R - k_N\,q_A \;=\; 0\,. \tag{63}$$

(62) implies that (63) holds if:

$$c_N - c_A - k_A\,q_A + k_N\,\frac{[\,a - \bar{p}\,]\,[\,b + k\,] - b\,[\,\bar{p} - c\,]}{b\,[\,b + k\,]} - k_N\,q_A \;=\; 0$$

$$\Leftrightarrow \;\; q_A\,[\,k_N + k_A\,] \;=\; c_N - c_A + k_N\,\frac{[\,a - \bar{p}\,]\,[\,b + k\,] - b\,[\,\bar{p} - c\,]}{b\,[\,b + k\,]}$$

$$\Leftrightarrow \;\; q_A \;=\; \frac{c_N - c_A}{k_N + k_A} + k_N\,\frac{[\,a - \bar{p}\,]\,[\,b + k\,] - b\,[\,\bar{p} - c\,]}{b\,[\,b + k\,]\,[\,k_N + k_A\,]}$$

$$= \frac{b\,[\,b+k\,]\,[\,c_N - c_A\,] + k_N\,[\,a-\bar{p}\,]\,[\,b+k\,] - b\,k_N\,[\,\bar{p}-c\,]}{b\,[\,b+k\,]\,[\,k_N + k_A\,]}\,. \tag{64}$$

(62) and (64) imply:

$$q_N \;=\; Q^R - q_A \;=\; \frac{[\,a-\bar{p}\,]\,[\,b+k\,] - b\,[\,\bar{p}-c\,]}{b\,[\,b\,+\,k\,]}$$

$$- \;\frac{b\,[\,b+k\,]\,[\,c_N - c_A\,] + k_N\,[\,a-\bar{p}\,]\,[\,b+k\,] - b\,k_N\,[\,\bar{p}-c\,]}{b\,[\,b+k\,]\,[\,k_N + k_A\,]}$$

$$= \frac{[\,a-\bar{p}\,]\,[\,b+k\,]\,[\,k_N+k_A\,] - b\,[\,\bar{p}-c\,]\,[\,k_N+k_A\,]}{b\,[\,b+k\,]\,[\,k_N + k_A\,]}$$

$$- \;\frac{b\,[\,b+k\,]\,[\,c_N - c_A\,] + k_N\,[\,a-\bar{p}\,]\,[\,b+k\,] - b\,k_N\,[\,\bar{p}-c\,]}{b\,[\,b+k\,]\,[\,k_N + k_A\,]}$$

$$= \frac{k_A\,[\,b+k\,]\,[\,a-\bar{p}\,] - b\,k_A\,[\,\bar{p}-c\,] \;-\; b\,[\,b+k\,]\,[\,c_N - c_A\,]}{b\,[\,b+k\,]\,[\,k_N + k_A\,]}\,. \tag{65}$$

(64) and (65) imply:

$$Q^R \;\equiv\; q_A \,+\, q_N \;=\; \frac{1}{b\,[\,b+k\,]\,[\,k_N+k_A\,]}\,\{\,[\,k_N+k_A\,]\,[\,b+k\,]\,[\,a-\bar{p}\,]$$

$$-\; b\,[\,k_N+k_A\,]\,[\,\bar{p}-c\,]\,\}$$

$$= \frac{[\,b+k\,]\,[\,a-\,\bar{p}\,\,] - b\,[\,\bar{p}-c\,]}{b\,[\,b+k\,]}\,. \tag{66}$$

(61) and (66) imply:

$$Q \;\equiv\; Q^R + q \;=\; \frac{[\,b+k\,]\,[\,a-\,\bar{p}\,\,] - b\,[\,\bar{p}-c\,]}{b\,[\,b+k\,]} + \frac{b\,[\,\bar{p}-c\,]}{b\,[\,b+k\,]} \;=\; \frac{a-\,\bar{p}}{b}\,.$$

(58) implies:

$$\frac{\partial^+\,\Pi_R}{\partial Q^R} \;=\; -\,b\,q_A + a - 2\,b\,Q^R - b\,q - c_N + b\,q_A - k_N\left[\,Q^R - q_A\,\right] - k^R\,Q^R$$

$$=\; a - 2\,b\,Q^R - b\,q - c_N - k_N\left[\,Q^R - q_A\,\right] - k^R\,Q^R$$

$$=\; \bar{p} - b\,Q^R - c_N - k_N\,q_N - k^R\,Q^R \;=\; \bar{p} - \left[\,b + k^R\,\right] Q^R - c_N - k_N\,q_N\,; \tag{67}$$

$$\frac{\partial^-\,\Pi_R}{\partial Q^R} \;=\; a - 2\,b\,Q^R - b\,q - c_N + b\,q_A - k_N\left[\,Q^R - q_A\,\right] - k^R\,Q^R$$

$$=\; a - 2\,b\,Q^R - b\,q - c_N + b\,q_A - k_N\,q_N - k^R\,Q^R$$

$$=\; \bar{p} - b\,Q^R - c_N + b\,q_A - k_N\,q_N - k^R\,Q^R$$

$$= \; \bar{p} - \left[b + k^R \right] Q^R - c_N + b\, q_A - k_N\, q_N \, . \tag{68}$$

(67) and (68) imply that (60) can be written as:

$$\bar{p} - \left[b + k^R \right] Q^R - c_N - k_N\, q_N \;\leq\; 0 \;<\; \bar{p} - \left[b + k^R \right] Q^R - c_N + b\, q_A - k_N\, q_N$$

$$\Leftrightarrow \quad \left[b + k^R \right] Q^R + c_N + k_N\, q_N - b\, q_A \;<\; \bar{p} \;\leq\; \left[b + k^R \right] Q^R + c_N + k_N\, q_N \, . \tag{69}$$

(62) and (65) imply:

$$\bar{p} \;\leq\; \left[b + k^R \right] Q^R + c_N + k_N\, q_N$$

$$\Leftrightarrow \quad \left[b + k^R \right] \frac{[\, a - \bar{p}\,]\,[\, b + k\,] \,-\, b\,[\,\bar{p} - c\,]}{b\,[\, b + k\,]} + c_N$$

$$+ \; k_N \, \frac{k_A\,[\, b + k\,]\,[\, a - \bar{p}\,] - b\, k_A\,[\,\bar{p} - c\,] - b\,[\, b + k\,]\,[\, c_N - c_A\,]}{b\,[\, b + k\,]\,[\, k_N + k_A\,]} \;\geq\; \bar{p}$$

$$\Leftrightarrow \quad \left[b + k^R \right] \frac{a\,[\, b + k\,] - \bar{p}\,[\, 2\, b + k\,] + b\, c}{b\,[\, b + k\,]} + c_N$$

$$+ \; k_N \, \frac{k_A\,[\, b + k\,]\, a - \bar{p}\, k_A\,[\, 2\, b + k\,] + b\, k_A\, c - b\,[\, b + k\,]\,[\, c_N - c_A\,]}{b\,[\, b + k\,]\,[\, k_N + k_A\,]} \;\geq\; \bar{p}$$

$$\Leftrightarrow \quad \left[b + k^R \right] \frac{a\,[\, b + \; k] + b\, c}{b\,[\, b + k\,]} + c_N + k_N \, \frac{k_A\,[\, b + k\,]\, a + b\, k_A\, c - b\,[\, b + k\,]\,[\, c_N - c_A\,]}{b\,[\, b + k\,]\,[\, k_N + k_A\,]}$$

$$\geq \; \bar{p} + \bar{p}\, \frac{k_N\, k_A\,[\, 2\, b + k\,]}{b\,[\, b + k\,]\,[\, k_N + k_A\,]} + \bar{p}\, \frac{[\, 2\, b + k\,]\left[\, b + k^R \right]}{b\,[\, b + k\,]}$$

$$\Leftrightarrow \quad \left[b + k^R \right] \frac{a\,[\, b + k\,] + b\, c}{b\,[\, b + k\,]} + c_N + k_N \, \frac{k_A\,[\, b + k\,]\, a + b\, k_A\, c - b\,[\, b + k\,]\,[\, c_N - c_A\,]}{b\,[\, b + k\,]\,[\, k_N + k_A\,]}$$

$$\geq \; \bar{p} \left[1 + \frac{k_N\, k_A\,[\, 2\, b + k\,]}{b\,[\, b + k\,]\,[\, k_N + k_A\,]} + \frac{[\, 2\, b + k]\left[b + k^R \right]}{b\,[\, b + k\,]} \right]$$

$$\Leftrightarrow \quad \left[b + k^R \right] [\, a\,(b + k) + b\, c\,]\,[\, k_N + k_A\,] + c_N\, b\,[\, b + k\,]\,[\, k_N + k_A\,]$$

$$+ \; k_N\,[\, k_A\,(b \, + \, k)\, a + b\, k_A\, c - b\,(b + k)\,(c_N - c_A)\,]$$

$$\geq \; \bar{p}\;[\, b\,(b + k)\,(k_N + k_A) + k_N\, k_A\,(2\, b + k)$$

$$+ \; (k_N + k_A)\,(2\, b + k)\left(b + k^R \right)] \; = \; \bar{p}\; D_3 \, . \tag{70}$$

The last equality in (70) reflects (8). (70) implies:

$$\bar{p} \;\leq\; \left[b + k^R \right] Q^R + c_N + k_N\, q_N$$

$$\Leftrightarrow \quad \bar{p} \;\leq\; \frac{1}{D_3} \{ \left[b + k^R \right] [a (b+k) + b c] [k_N + k_A] + c_N \, b [b+k] [k_N + k_A]$$

$$+ \;\; k_N [k_A (b+k) \, a + b \, k_A \, c - b (b+k) (c_N - c_A)] \}$$

$$\Leftrightarrow \quad \bar{p} \;\leq\; \frac{1}{D_3} \{ \left[b + k^R \right] [a (b+k) + b c] [k_N + k_A] + c_N \, b [b+k] [k_N + k_A]$$

$$+ \;\; k_N \, k_A [b+k] \, a + b \, k_N \, k_A \, c - k_N \, b [b+k] [c_N - c_A] \}$$

$$\Leftrightarrow \quad \bar{p} \;\leq\; \frac{1}{D_3} \{ \left[b + k^R \right] [a (b+k) + b c] [k_N + k_A] + c_N \, b [b+k] \, k_A$$

$$+ \, k_N \, k_A [b+k] \, a + b \, k_A \, k_N \, c + k_N \, b [b+k] \, c_A$$

$$+ \, k_N \, k_A [b+k] \, a + b \, k_A \, k_N \, c + k_N \, b [b+k] \, c_A \}$$

$$\Leftrightarrow \quad \bar{p} \;\leq\; \frac{1}{D_3} \{ a [b+k] \left[\left(b + k^R \right) (k_N + k_A) + k_N \, k_A \right]$$

$$+ \;\; c \left[b (k_N + k_A) \left(b + k^R \right) + b \, k_A \, k_N \right] + c_N \, b [b+k] \, k_A + k_N \, b [b+k] \, c_A \}$$

$$\Leftrightarrow \quad \bar{p} \;\leq\; \frac{1}{D_3} \{ a [b+k] \left[\left(b + k^R \right) (k_N + k_A) + k_N \, k_A \right]$$

$$+ \;\; b \, c \left[(k_N + k_A) \left(b + k^R \right) + k_A \, k_N \right] + c_N \, b [b+k] \, k_A + k_N \, b [b+k] \, c_A \}$$

$$\Leftrightarrow \quad \bar{p} \;\leq\; \bar{p}_3 \, . \tag{71}$$

(62), (64), and (65) imply:

$$\left[b + k^R \right] Q^R + c_N + k_N \, q_N - b \, q_A \;<\; \bar{p}$$

$$\Leftrightarrow \quad \left[b + k^R \right] \; \frac{[a - \bar{p}] [b+k] - b [\bar{p} - c]}{b [b+k]} + c_N$$

$$+ \; k_N \; \frac{k_A [b+k] [a - \bar{p}] - b \, k_A [\bar{p} - c] - b [b+k] [c_N - c_A]}{b [b+k] [k_N \, + \, k_A]}$$

$$- \; b \; \frac{b [b+k] [c_N - c_A] + k_N [a - \bar{p}] [b+k] - b \, k_N [\bar{p} - c]}{b [b+k] [k_N \, + \, k_A]} \;<\; \bar{p}$$

$$\Leftrightarrow \quad \left[b + k^R \right] \; \frac{a [b+k] - \bar{p} \, [2 \, b + k] + b \, c}{b [b+k]} + c_N$$

$$+ \; k_N \; \frac{k_A [b+k] \, a - \bar{p} \, k_A [2 \, b + k] + b \, k_A \, c - b [b+k] [c_N - c_A]}{b [b+k] [k_N + k_A]}$$

$$- \ b \ \frac{b\,[\,b+k\,]\,[\,c_N - c_A\,] + k_N\,a\,[\,b+k\,] - \bar{p}\;k_N\,[\,2\,b+k\,] + b\,k_N\,c}{b\,[\,b+k\,]\,[\,k_N + k_A\,]} \ < \ \bar{p}$$

$$\Leftrightarrow \quad \left[\,b+k^R\,\right] \ \frac{a\,[\,b+k\,] + b\,c}{b\,[b\,+\,k]} + c_N + k_N \ \frac{k_A\,[\,b+k\,]\,a + b\,k_A\,c - b\,[\,b+k\,]\,[\,c_N - c_A\,]}{b\,[\,b+k\,]\,[\,k_N + k_A\,]}$$

$$- \ b \ \frac{b\,[\,b+k\,]\,[\,c_N - c_A\,] + k_N\,a\,[\,b+k\,] + b\,k_N\,c}{b\,[\,b+k\,]\,[\,k_N + k_A\,]}$$

$$< \ \bar{p} + \bar{p} \ \frac{[\,k_N\,k_A - k_N\,b\,]\,[\,2\,b+k\,]}{b\,[\,b+k\,]\,[\,k_N + k_A\,]} + \bar{p} \ \frac{[\,2\,b+k\,]\left[\,b+k^R\,\right]}{b\,[\,b+k\,]}$$

$$\Leftrightarrow \quad \left[\,b+k^R\,\right] \ \frac{a\,[\,b+k\,] + b\,c}{b\,[\,b+k\,]} + c_N + k_N \ \frac{k_A\,[\,b+k\,]\,a + b\,k_A\,c - b\,[\,b+k\,]\,[\,c_N - c_A\,]}{b\,[\,b+k\,]\,[k_N \ + \ k_A]}$$

$$- \ b \ \frac{b\,[\,b+k\,]\,[\,c_N - c_A\,] + k_N\,a\,[\,b+k\,]\ b\,k_N\,c}{b\,[\,b+k\,]\,[\,k_N + k_A\,]}$$

$$< \ \bar{p}\left[1 + \frac{[\,k_N\,k_A - k_N\,b\,]\,[\,2\,b+k\,]}{b\,[\,b+k\,]\,[\,k_N + k_A\,]} + \frac{[\,2\,b+k\,]\left[\,b+k^R\,\right]}{b\,[\,b+k\,]}\right]$$

$$\Leftrightarrow \quad \left[\,b+k^R\,\right]\,[\,a\,(b+k) + b\,c\,]\,[\,k_N + k_A\,] + c_N\,b\,[\,b+k\,]\,[\,k_N + k_A\,]$$

$$+ \ k_N\,[\,k_A\,(b+k)\,a + b\,k_A\,c - b\,(b+k)\,(c_N - c_A)\,]$$

$$- \ b\,[\,b\,(b+k)\,(c_N - c_A) + k_N\,a\,(b+k) + b\,k_N\,c\,]$$

$$< \ \bar{p}\ [\,b\,(b+k)\,(k_N + k_A) + k_N\,(k_A - b)\,(2\,b+k)$$

$$+ \ (k_N + k_A)\,(2\,b+k)\left(b+k^R\right)] \ = \ \bar{p}\ D_2\,. \tag{72}$$

The last equality in (72) reflects (7). (7) and (72) imply:

$$\left[\,b+k^R\,\right] Q^R + c_N + k_N\,q_N - b\,q_A \ < \ \bar{p}$$

$$\Leftrightarrow \ \ \bar{p} \ > \ \frac{1}{D_2}\,\{\,\left[\,b+k^R\,\right]\,[\,a\,(b+k) + b\,c\,]\,[\,k_N + k_A\,] + c_N\,b\,[\,b+k\,]\,[\,k_N + k_A\,]$$

$$+ \ \ k_N\,[\,k_A\,(b+k)\,a + b\,k_A\,c - b\,(b+k)\,(c_N - c_A)\,]$$

$$- \ \ b\,[\,b\,(b+k)\,(c_N - c_A) + k_N\,a\,(b+k) + b\,k_N\,c\,]\,\}$$

$$\Leftrightarrow \ \ \bar{p} \ > \ \frac{1}{D_2}\,\{\,a\left[\,\left(b+k^R\right)(b+k)\,(k_N + k_A) + k_N\,k_A\,(b+k) - b\,(b+k)\,k_N\,\right]$$

$$+ \ \ c\left[\,b\,(k_N + k_A)\left(b+k^R\right) + b\,k_A\,k_N - b^2 k_N\,\right]$$

$$+ \ \ c_N\,b\,[\,b+k\,]\,[\,k_A - b\,] + b\,[\,k_N + b\,]\,[\,b+k\,]\,c_A\,\}$$

$$\Leftrightarrow \quad \bar{p} \; > \; \frac{1}{D_2} \{ a [b + k] \left[\left(b + k^R \right) (k_N + k_A) + k_N \, k_A - b \, k_N \right]$$

$$+ \; c \, b \left[(k_N + k_A) \left(b + k^R \right) + k_A \, k_N - b \, k_N \right]$$

$$+ \; c_N \, b [b + k] [k_A - b] + b [k_N + b] [b + k] \, c_A \} \; = \; \bar{p}_2 \, .$$

(7), (8), (67), (68), and (71) imply:

$$\bar{p}_2 \;=\; \left[b + k^R \right] Q^R + c_N + k_N \, q_N - b \, q_A \quad \text{and}$$

$$\bar{p}_3 \;=\; \left[b + k^R \right] Q^R + c_N + k_N \, q_N \, . \tag{73}$$

(73) implies that $\bar{p}_2 < \bar{p}_3$ because $q_A > 0$ when $\bar{p} > \bar{p}_1$. □

Lemma A4. *Suppose $\bar{p} > \bar{p}_3$. Then in equilibrium*:

$$q_A \;=\; \frac{1}{D_3} \{ [a - c_A] \left[2 \, b \, k + 2 \, b \, k_N + 2 \, b \, k^R + k \, k_N + k \, k^R + 3 \, b^2 \right]$$

$$- \; [a - c_N] \left[2 \, b \, k + 2 \, b \, k^R + k \, k^R + 3 \, b^2 \right] - b \, k_N [a - c] \} \, ; \tag{74}$$

$$q_N \;=\; \frac{1}{D_3} \{ [a - c_N] \left[2 \, b \, k + 2 \, b \, k_A + 2 \, b \, k^R + k \, k_A + k \, k^R + 3 \, b^2 \right]$$

$$- \; [a - c_A] \left[2 \, b \, k + 2 \, b \, k^R + k \, k^R + 3 \, b^2 \right] - b \, k_A [a - c] \} \, ; \tag{75}$$

$$q \;=\; \frac{1}{D_3} \{ [a - c] \left[2 \, b \, k_A + 2 \, b \, k_N + k_A \, k_N + k_A \, k^R + k_N \, k^R \right]$$

$$- \; b \, k_A [a - c_N] - b \, k_N [a - c_A] \} \, ; \quad \text{and} \tag{76}$$

$$Q^R \;\equiv\; q_A + q_N \;=\; \frac{1}{D_3} \{ [a - c_A] \, k_N [2 \, b + k] + [a - c_N] \, k_A [2 \, b + k]$$

$$- \; b [k_A + k_N] [a - c] \} \tag{77}$$

where D_3 is as specified in (8).

Proof. (4) implies that when the price cap does not bind, [P-R] is:

$$\underset{q_A, \, q_N}{\text{Maximize}} \quad [a - b (q_A + q_N + q)] [q_A + q_N] - c_A \, q_A - \frac{k_A}{2} [q_A]^2$$

$$- \; c_N \, q_N - \frac{k_N}{2} [q_N]^2 - \frac{k^R}{2} [q_A + q_N]^2 \, . \tag{78}$$

Differentiating (78) with respect to q_A provides:

$$a - b [q_A + q_N + q] - b [q_A + q_N] - c_A - k_A \, q_A - k^R [q_A + q_N] \;=\; 0$$

$$\Leftrightarrow \quad a - b [q_N + q] - b \, q_N - c_A - k^R \, q_N \;=\; q_A \left[2 \, b + k_A + k^R \right]$$

$$\Leftrightarrow \quad q_A \;=\; \frac{a - c_A - \left[\,2\,b + k^R\,\right] q_N - b\,q}{2\,b + k_A + k^R}\,. \tag{79}$$

Corresponding differentiation of (78) with respect to q_N provides:

$$q_N \;=\; \frac{a - c_N - \left[\,2\,b + k^R\,\right] q_A - b\,q}{2\,b + k_N + k^R}\,. \tag{80}$$

(32) implies:

$$q \;=\; \frac{a - c}{2\,b + k} - \frac{b}{2\,b + k}\,[\,q_A + q_N\,]\,. \tag{81}$$

<u>Definitions</u>. $K_A \;\equiv\; 2\,b + k_A + k^R \quad$ and $\quad K_N \;\equiv\; 2\,b + k_N + k^R\,.$ (82)

(79), (81), and (82) imply:

$$q_A \;=\; \frac{a - c_A}{K_A} - \frac{\left[\,2\,b + k^R\,\right] q_N}{K_A} - \frac{b}{K_A}\left[\frac{a - c - b\,(q_A + q_N)}{2\,b + k}\right]$$

$$\Rightarrow \quad q_A\left[1 - \frac{b^2}{[\,2\,b + k\,]\,K_A}\right]$$

$$= \frac{[\,2\,b + k\,]\,[\,a - c_A\,] - \left[\,2\,b + k^R\,\right][\,2\,b + k\,]\,q_N - b\,[\,a - c\,] + b^2\,q_N}{[\,2\,b + k\,]\,K_A}$$

$$\Rightarrow \quad q_A\left[\frac{[\,2\,b + k\,]\,K_A - b^2}{[\,2\,b + k\,]\,K_A}\right]$$

$$= \frac{[\,2\,b + k\,]\,[\,a - c_A\,] - b\,[\,a - c\,] - \left(\left[\,2\,b + k^R\,\right][\,2\,b + k\,] - b^2\right) q_N}{[\,2\,b + k\,]\,K_A}$$

$$\Rightarrow \quad q_A \;=\; \frac{[\,2\,b + k\,]\,[\,a - c_A\,] - b\,[\,a - c\,]}{D_A} - \frac{B}{D_A}\,q_N$$

where $D_A \;\equiv\; [\,2\,b + k\,]\,K_A - b^2 \quad$ and $\quad B \;\equiv\; \left[\,2\,b + k^R\,\right][\,2\,b + k\,] - b^2\,.$ (83)

(80) – (82) imply:

$$q_N \;=\; \frac{a - c_N}{K_N} - \frac{\left[\,2\,b + k^R\,\right] q_A}{K_N} - \frac{b}{K_N}\left[\frac{a - c - b\,(q_A + q_N)}{2\,b + k}\right]$$

$$\Rightarrow \quad q_N\left[1 - \frac{b^2}{[\,2\,b + k\,]\,K_N}\right]$$

$$= \frac{[\,2\,b + k\,]\,[\,a - c_N\,] - \left[\,2\,b + k^R\,\right][\,2\,b + k\,]\,q_A - b\,[\,a - c\,] + b^2\,q_A}{[\,2\,b + k\,]\,K_N}$$

$$\Rightarrow \quad q_N\left[\frac{[\,2\,b + k\,]\,K_N - b^2}{[\,2\,b + k\,]\,K_N}\right]$$

$$= \frac{[\,2\,b+k\,]\,[\,a-c_N\,]-b\,[\,a-c\,]-\left(\left[\,2\,b+k^R\,\right][\,2\,b+k\,]-b^2\right)q_A}{[\,2\,b+k\,]\,K_N}$$

$$\Rightarrow \quad q_N \;=\; \frac{[\,2\,b+k\,]\,[\,a-c_N\,]-b\,[\,a-c\,]}{D_N} - \frac{B}{D_N}\,q_A$$

$$\text{where} \quad K_N \;\equiv\; 2\,b+k_N+k^R \quad \text{and} \quad D_N \;\equiv\; [\,2\,b+k\,]\,K_N-b^2\,. \tag{84}$$

(83) and (84) imply:

$$q_A \;=\; \frac{[\,2\,b+k\,]\,[\,a-c_A\,]-b\,[\,a-c\,]}{D_A} - \frac{B}{D_A\,D_N}\,\{\,[\,2\,b+k\,]\,[\,a-c_N\,]-b\,[\,a-c\,]-B\,q_A\,\}$$

$$\Rightarrow \quad q_A\left[1-\frac{B^2}{D_A\,D_N}\right] \;=\; \frac{1}{D_A\,D_N}\Big\{\,[\,2\,b+k\,]\,D_N\,[\,a-c_A\,]-b\,D_N\,[\,a-c\,]$$

$$-\;B\,[\,2\,b+k\,]\,D_N\,[\,a-c_N\,]+b\,B\,[\,a-c\,]\,\Big\}$$

$$\Rightarrow \quad q_A\left[\,D_A\,D_N-B^2\,\right] \;=\; [\,2\,b+k\,]\,D_N\,[\,a-c_A\,]+b\,[\,B-D_N\,]\,[\,a-c\,]$$

$$+\;[\,2\,b+k\,]\,B\,[\,a-c_N\,]\,. \tag{85}$$

(83) and (84) imply:

$$D_A\,D_N-B^2 \;=\; \left[\,(2\,b+k)\,K_A-b^2\,\right]\left[\,(2\,b+k)\,K_N-b^2\,\right]-\left[\,\left(2\,b+k^R\right)(2\,b+k)-b^2\,\right]^2$$

$$=\; [\,2\,b+k\,]^2\;K_A\,K_N-b^2\,[\,2\,b+k\,]\,K_A-b^2\,[\,2\,b+k\,]\,K_N+b^4$$

$$-\;[\,2\,b+k\,]^2\left[\,2\,b+k^R\,\right]^2+2\,b^2\,[\,2\,b+k\,]\left[\,2\,b+k^R\,\right]-b^4$$

$$=\; [\,2\,b+k\,]\;\{\,[\,2\,b+k\,]\,K_A\,K_N-b^2\,[\,K_A+K_N\,]+2\,b^2\left[\,2\,b+k^R\,\right]$$

$$-\;[\,2\,b+k\,]\left[\,2\,b+k^R\,\right]^2\}\,. \tag{86}$$

(82) implies that the term in $\{\cdot\}$ in (86) is:

$$[\,2\,b+k\,]\left[\,2\,b+k^R+k_A\,\right]\left[\,2\,b+k^R+k_N\,\right]-b^2\left[\,4\,b+k_A+k_N+2\,k^R\,\right]$$

$$+\;2\,b^2\left[\,2\,b+k^R\,\right]-[\,2\,b+k\,]\left[\,2\,b+k^R\,\right]^2$$

$$=\; [\,2\,b+k\,]\left\{\,\left[\,2\,b+k^R\,\right]^2+[\,k_A+k_N\,]\left[\,2\,b+k^R\,\right]+k_A\,k_N\,\right\}$$

$$+\;2\,b^2\left[\,2\,b+k^R\,\right]-[\,2\,b+k\,]\left[\,2\,b+k^R\,\right]^2-b^2\left[\,2\left(2\,b+k^R\right)+k_A+k_N\,\right]$$

$$=\; \left[\,2\,b+k^R\,\right]\left\{\,[\,2\,b+k\,]\,[\,k_A+k_N\,]+2\,b^2-2\,b^2\,\right\}+[\,2\,b+k\,]\,k_A\,k_N$$

$$+\;[\,2\,b+k\,]\,k_A\,k_N-b^2\,[\,k_A+k_N\,]$$

$$= [\, k_A + k_N \,] \left\{ [\, 2\, b + k \,] \left[2\, b + k^R \right] - b^2 \right\} + [\, 2\, b + k \,]\, k_A\, k_N$$

$$= [\, k_A + k_N \,] \left\{ [\, 2\, b + k \,] \left[b + k^R \right] + b\, [\, 2\, b + k \,] - b^2 \right\} + [\, 2\, b + k \,]\, k_A\, k_N$$

$$= [\, k_A + k_N \,] \left\{ [\, 2\, b + k \,] \left[b + k^R \right] + b\, [\, b + k \,] \right\} + [\, 2\, b + k \,]\, k_A\, k_N \;=\; D_3\, . \tag{87}$$

The last equality in (87) reflects (8).

(82) and (84) imply:

$$D_N \;=\; [\, 2\, b + k \,] \left[2\, b + k_N + k^R \right] - b^2 \;=\; 2\, b\, [\, 2\, b + k \,] - b^2 + [\, 2\, b + k \,] \left[k_N + k^R \right]$$

$$= 3\, b^2 + 2\, b\, k + [\, 2\, b + k \,] \left[k_N + k^R \right] . \tag{88}$$

(82) and (84) imply:

$$B - D_N \;=\; [\, 2\, b + k \,] \left[2\, b + k^R \right] - b^2 - \left\{ [\, 2\, b + k \,] \left[2\, b + k_N + k^R \right] - b^2 \right\}$$

$$= [\, 2\, b + k \,] \left[2\, b + k^R - 2\, b - k_N - k^R \right] \;=\; -\, [\, 2\, b + k \,]\, k_N\, . \tag{89}$$

(83) and (85) – (89) imply that (74) holds. Furthermore, (74) and the symmetry of q_A and q_N in the analysis imply that (75) holds.

Observe that:

$$3\, b^2 + 2\, b\, k + [\, 2\, b + k \,] \left[k_N + k^R \right] - \left[(2\, b + k) \left(2\, b + k^R \right) - b^2 \right]$$

$$= 4\, b^2 + 2\, b\, k + [\, 2\, b + k \,] \left[k_N + k^R - \left(2\, b + k^R \right) \right]$$

$$= 2\, b\, [\, 2\, b + k \,] + [\, 2\, b + k \,]\, [\, k_N - 2\, b \,] \;=\; [\, 2\, b + k \,]\, k_N\, ; \;\; \text{and}$$

$$3\, b^2 + 2\, b\, k + [\, 2\, b + k \,] \left[k_A + k^R \right] - \left[(2\, b + k) \left(2\, b + k^R \right) - b^2 \right]$$

$$= 4\, b^2 + 2\, b\, k + [\, 2\, b + k \,] \left[k_A + k^R - \left(2\, b + k^R \right) \right]$$

$$= 2\, b\, [\, 2\, b + k \,] + [\, 2\, b + k \,]\, [\, k_A - 2\, b \,] \;=\; [\, 2\, b + k \,]\, k_A\, . \tag{90}$$

(74), (75), and (90) imply that $Q^R = q_A + q_N$ is as specified in (77).

(77) and (81) imply:

$$q \;=\; \frac{[\, a - c \,]\, D_3}{[\, 2\, b + k \,]\, D_3}$$

$$-\; \frac{b}{[\, 2\, b + k \,]\, D_3} \{\, [\, a - c_A \,]\, k_N\, [\, 2\, b + k \,] + [\, a \, - c_N \,]\, k_A\, [\, 2\, b + k \,]$$

$$-\; b\, [\, k_A + k_N \,]\, [\, a - c \,] \,\}$$

$$= \; \frac{1}{[\, 2\, b + k \,]\, D_3} \{\, [\, a - c \,] \left[D_3 + b^2\, (k_A + k_N) \right] - [\, 2\, b + k \,]\, b\, k_A\, [\, a \, - c_N \,]$$

$$- \left[2\,b + k \right] b\, k_N \left[a \, - c_A \right] \Big\} . \tag{91}$$

(8) implies:

$$\begin{aligned}
D_3 + b^2 \left[k_A + k_N \right] \;&=\; \left[2\,b + k \right] k_A\, k_N + \left[k_A + k_N \right] \left[b^2 + b\,(b+k) + (2\,b+k)\,\left(b + k^R \right) \right] \\
&=\; \left[2\,b + k \right] k_A\, k_N + \left[k_A + k_N \right] \left[2\,b^2 + b\,k + 2\,b^2 + 2\,b\,k^R + b\,k + k\,k^R \right] \\
&=\; \left[2\,b + k \right] k_A\, k_N + \left[k_A + k_N \right] \left[4\,b^2 + 2\,b\,k + 2\,b\,k^R + k\,k^R \right] \\
&=\; \left[2\,b + k \right] k_A\, k_N + \left[k_A + k_N \right] \left[2\,b\,(2\,b+k) + k^R\,(2\,b+k) \right] \\
&=\; \left[2\,b + k \right] \left\{ k_A\, k_N + \left[k_A + k_N \right] \left[2\,b + k^R \right] \right\} . \qquad (92)
\end{aligned}$$

(91) and (92) imply that q is as specified in (76).

(74) – (76) imply:

$$\begin{aligned}
P(Q) \;&=\; a - b \left[q_A + q_N + q \right] \\
&=\; a - \frac{b}{D_3} \left[B_1\,(a - c_A) + B_2\,(a - c_N) + B_3\,(a - c\,) \right] \qquad (93)
\end{aligned}$$

where $B_1 \;=\; k_N \left[b + k \right];\;\; B_2 \;=\; k_A \left[b + k \right];\;$ and

$$B_3 \;=\; \left[b + k^R \right] \left[k_A + k_N \right] + k_A\, k_N\,. \tag{94}$$

(93) implies:

$$P(Q) \;=\; \frac{\left[D_3 - b\,(B_1 + B_2 + B_3) \right] a + b\,B_1\,c_A + b\,B_2\,c_N + b\,B_3\,c}{D_3}. \tag{95}$$

(94) implies:

$$\begin{aligned}
B_1 + B_2 + B_3 \;&=\; \left[k_A + k_N \right] \left[b + k \right] + \left[b + k^R \right] \left[k_A + k_N \right] + k_A\, k_N \\
&=\; \left[2\,b + k + k^R \right] \left[k_A + k_N \right] + k_A\, k_N.
\end{aligned}$$

(8) and (94) imply:

$$\begin{aligned}
&D_3 - b \left[B_1 + B_2 + B_3 \right] \\
&=\; b \left[b + k \right] \left[k_N + k_A \right] + k_N\, k_A \left[2\,b + k \right] + \left[k_N + k_A \right] \left[2\,b + k \right] \left[b + k^R \right] \\
&\qquad - b \left[2\,b + k + k^R \right] \left[k_A + k_N \right] - b\, k_N\, k_A \\
&=\; b \left[b + k \right] \left[k_N + k_A \right] + k_N\, k_A \left[2\,b + k \right] + \left[k_N + k_A \right] \left[2\,b^2 + k\,b + 2\,k^R b + k\,k^R \right] \\
&\qquad - \left[2\,b^2 + k\,b + b\,k^R \right] \left[k_A + k_N \right] - b\, k_N\, k_A \\
&=\; b \left[b + k \right] \left[k_N + k_A \right] + k_N\, k_A \left[b + k \right] + \left[k_N + k_A \right] \left[b\,k^R + k\,k^R \right] \\
&=\; b \left[b + k \right] \left[k_N + k_A \right] + k_N\, k_A \left[b + k \right] + k^R \left[k_N + k_A \right] \left[b + k \right]
\end{aligned}$$

$$= \left[\, b + k \,\right] \left[\, b \,(k_N + k_A) + k_N \, k_A + k^R \,(k_N + k_A) \,\right]$$

$$= \left[\, b + k \,\right] \left[\, \left(b + k^R\right) (k_N + k_A) + k_N \, k_A \,\right] . \tag{96}$$

(94), (95), and (96) imply that the price cap does not bind if:

$$\bar{p} \;>\; \frac{a \left[\, D_3 - b \,(B_1 + B_2 + B_3) \,\right] + b \, B_1 \, c_A + b \, B_2 \, c_N + b \, B_3 \, c}{D_3}$$

$$= \frac{1}{D_3} \left\{ a \left[\, b + k \,\right] \left[\, \left(b + k^R\right) (k_N + k_A) + k_N \, k_A \,\right] + b \left[\, b + k \,\right] k_N \, c_A \right.$$

$$\left. + \;\; b \left[\, b + k \,\right] k_A \, c_N + b \, c \left[\, \left(b + k^R\right) (k_A + k_N) + k_A \, k_N \,\right] \right\}$$

$$= \frac{1}{D_3} \left\{ \left[\, (b + k) \, a + b \, c \,\right] \left[\, \left(b + k^R\right) (k_N + k_A) + k_N \, k_A \,\right] \right.$$

$$\left. + \;\; b \left[\, b + k \,\right] k_N \, c_A + b \left[\, b + k \,\right] k_A \, c_N \right\} \;=\; \bar{p}_3 \, . \tag{97}$$

The last equality in (97) reflects (8). $\square$

Definitions

$q_{A1}(\bar{p}_1)$, $q_{N1}(\bar{p}_1)$, and $q_1(\bar{p}_1)$, respectively, denote the values of q_A, q_N, and q specified in Lemma A1, where $\bar{p} \leq \bar{p}_1$.

$q_{A2}(\bar{p}_1)$, $q_{N2}(\bar{p}_1)$, and $q_2(\bar{p}_1)$, respectively, denote the values of q_A, q_N, and q specified in Lemma A2, where $\bar{p} \in (\, \bar{p}_1, \bar{p}_2 \,]$.

Lemma A5. $\lim_{\bar{p} \to \bar{p}_1} q_{A2}(\bar{p}) \;=\; q_{A1}(\bar{p}_1)$, $\lim_{\bar{p} \to \bar{p}_1} q_{N2}(\bar{p}) \;=\; q_{N1}(\bar{p}_1)$, *and* $\lim_{\bar{p} \to \bar{p}_1} q_2(\bar{p}) \;=\; q_1(\bar{p}_1)$.

Proof. (11), (12), and (19) imply that when $\bar{p} \leq \bar{p}_1$, q_N, q, and q_A are determined by:

$$\frac{\partial \pi^R}{\partial q_N} \;=\; a - 2 \, b \, q_N - b \, q - c_N - k_N \, q_N - k^R \, q_N \;=\; 0 \, ;$$

$$\frac{\partial \pi}{\partial q} \;=\; a - b \, q_N - 2 \, b \, q - c - k \, q \;=\; 0 \, ;$$

$$q_A \;=\; 0; \;\; \text{and} \;\; \frac{\partial \pi^R}{\partial q_A} \;=\; \bar{p} - c_A - b \, q_N - k^R \, q_N \;\leq\; 0 \, . \tag{98}$$

(19) implies that the weak inequality in (98) holds as an equality when $\bar{p} = \bar{p}_1$.

(25), (26), and (32) imply that when $\bar{p} \in (\, \bar{p}_1, \bar{p}_2 \,]$, q_N, q, and q_A are determined by:

$$\frac{\partial \pi^R}{\partial q_N} \;=\; a - 2 \, b \, q_N - b \, q - b \, q_A - c_N - k_N \, q_N - k^R \left[\, q_N + q_A \,\right] \;=\; 0 \, ;$$

$$\frac{\partial \pi}{\partial q} \;=\; a - b \, q_N - b \, q_A - 2 \, b \, q - c - k \, q \;=\; 0 \, ;$$

$$\frac{\partial \pi^R}{\partial q_A} \;=\; \bar{p} - c_A - k_A\, q_A - b\, q_N - k^R\, [\, q_N + q_A \,] \;=\; 0\,. \tag{99}$$

(6) and (25) imply:

$$\begin{aligned}
\lim_{\bar{p} \to \bar{p}_1} q_{A2}(\bar{p}) \;&=\; \frac{1}{D} \{ \left[3\, b^2 + 2\, b \left(k + k_N + k^R \right) + k \left(k_N + k^R \right) \right] [\, \bar{p}_1 - c_A \,] \\
&\qquad + \; b \left[b + k^R \right] [\, a - c \,] - [\, 2\, b + k \,] \left[b + k^R \right] [\, a - c_N \,] \} \\
&=\; \frac{1}{D} \{ \left[3\, b^2 + 2\, b \left(k + k_N + k^R \right) + k \left(k_N + k^R \right) \right] \\
&\qquad \cdot \left[b + k^R \right] \frac{[\, a - c_N \,]\, [\, 2\, b + k \,] \; - b\, [\, a - c \,]}{[\, 2\, b + k_N + k^R \,]\, [\, 2\, b + k \,] - b^2} \\
&\qquad + \; b \left[b + k^R \right] [\, a - c \,] - [\, 2\, b + k \,] \left[b + k^R \right] [\, a - c_N \,] \} \\
&=\; \frac{1}{D} \{ \left[b + k^R \right] \; [\, a - c_N \,]\, [\, 2\, b + k \,] \; - b \left[b + k^R \right] [\, a - c \,] \\
&\qquad + \; b \left[b + k^R \right] [\, a - c \,] - [\, 2\, b + k \,] \left[b + k^R \right] [\, a - c_N \,] \} \;=\; 0\,.
\end{aligned} \tag{100}$$

(100) reflects the fact that:

$$\begin{aligned}
\left[2\, b + k_N + k^R \right] [\, 2\, b + k \,] - b^2 \;&=\; 3\, b^2 + 2\, b\, k + \left[k_N^R + k^R \right] [\, 2\, b + k \,] \\
&=\; 3\, b^2 + 2\, b \left[k + k_N + k^R \right] + k \left[k_N + k^R \right].
\end{aligned}$$

(100) implies that $\lim_{\bar{p} \to \bar{p}_1} q_{A2}(\bar{p}) \;=\; q_{A1}(\bar{p}_1)$. The equations in (99) coincide with the equations in (98) when $\bar{p} \;=\; \bar{p}_1$. Therefore, because (20), (21), and (23) imply that q_A, q_N, and q are continuous functions of $\bar{p}$, $\lim_{\bar{p} \to \bar{p}_1} q_{A2}(\bar{p}) \;=\; q_{A1}(\bar{p}_1)$, $\lim_{\bar{p} \to \bar{p}_1} q_{N2}(\bar{p}) \;=\; q_{N1}(\bar{p}_1)$, and $\lim_{\bar{p} \to \bar{p}_1} q_2(\bar{p}) \;=\; q_1(\bar{p}_1)$. $\square$

Lemma A6. $0 < \bar{p}_1 < \bar{p}_2 < \bar{p}_3\,.$

Proof. The proof of Lemma A3 establishes that $\bar{p}_2 \;<\; \bar{p}_3$. From (6):

$$\bar{p}_1 \;=\; \frac{[\, a - c_N \,]\, [\, 2\, b + k \,] - b\, [\, a - c \,]}{[\, 2\, b + k_N + k^R \,]\, [\, 2\, b + k \,] - b^2} \left[b + k^R \right] + c_A \;>\; 0\,. \tag{101}$$

The inequality in (101) holds because $[\, a - c_N \,]\, [\, 2\, b + k \,] \, - b\, [\, a - c \,] \;>\; 0$, from (3).

To prove that $\bar{p}_1 \;<\; \bar{p}_2$, let $Q_1(\bar{p})$ denote the value of $Q(\bar{p})$ specified in Lemma A1, and let $Q_2(\bar{p})$ denote the value of $Q(\bar{p})$ specified in Lemma A2. Lemma A5 implies:

$$Q_1(\bar{p}_1) \;=\; Q_2(\bar{p}_1)\,. \tag{102}$$

Lemma A2 implies:

$$\bar{p} \;<\; P(\, Q_2(\bar{p})\,) \;\;\Leftrightarrow\;\; \bar{p} \;<\; \bar{p}_2\,. \tag{103}$$

(102) and (103) imply that if $\bar{p}_1 < P(Q_1(\bar{p}_1))$, then:

$$\bar{p}_1 < P(Q_2(\bar{p}_1)) \Leftrightarrow \bar{p}_1 < \bar{p}_2 . \tag{104}$$

The first inequality in (104) holds because (102) implies that $P(Q_1(\bar{p}_1)) = P(Q_2(\bar{p}_1))$. The equivalence in (104) reflects (103). (104) implies that to establish that $\bar{p}_1 < \bar{p}_2$, it suffices to show that $\bar{p}_1 < P(Q_1(\bar{p}_1))$.

(6) implies:

$$\begin{aligned}
\bar{p}_1 &= c_A + \frac{[a - c_N][2b + k] - b[a - c]}{[2b + k_N + k^R][2b + k] - b^2} \left[b + k^R \right] \\
&= \frac{1}{[2b + k_N + k^R][2b + k] - b^2} \\
&\quad \cdot \left\{ c_A \left[\left(2b + k_N + k^R\right)(2b + k) - b^2 \right] + [a - c_N][2b + k]\left[b + k^R \right] \right. \\
&\qquad \left. - b\left[b + k^R \right][a - c] \right\} .
\end{aligned} \tag{105}$$

Recall from (15) that when $q_A = 0$ and the price cap binds, the equilibrium price is:

$$\begin{aligned}
P(Q) &= a - b[q + q_N] = a - b \, \frac{[a - c]\left[b + k_N + k^R \right] + [a - c_N][b + k]}{[2b + k_N + k^R][2b + k] - b^2} \\
&= \frac{1}{[2b + k_N + k^R][2b + k] - b^2} \\
&\quad \cdot \left\{ a \left[\left(2b + k_N + k^R\right)(2b + k) - b^2 \right] - b[a - c]\left[b + k_N + k^R \right] \right. \\
&\qquad \left. - b[a - c_N][b + k] \right\} .
\end{aligned} \tag{106}$$

(105) and (106) imply that $\bar{p}_1 < P(Q)$ if:

$$\begin{aligned}
& a \left[\left(2b + k_N + k^R\right)(2b + k) - b^2 \right] - b[a - c]\left[b + k_N + k^R \right] - b[a - c_N][b + k] \\
&\quad > c_A \left[\left(2b + k_N + k^R\right)(2b + k) - b^2 \right] + [a - c_N][2b + k]\left[b + k^R \right] \\
&\qquad - b\left[b + k^R \right][a - c] \\
\Leftrightarrow \quad & [a - c_A]\left[\left(2b + k_N + k^R\right)(2b + k) - b^2 \right] - b[a - c]k_N \\
&\quad - [a - c_N]\left[(b + k)b + (2b + k)\left(b + k^R\right) \right] > 0 \\
\Leftrightarrow \quad & [a - c_A]\left[2bk + 2bk_N + 2bk^R + kk_N + kk^R + 3b^2 \right] \\
&\quad - bk_N[a - c] - [a - c_N]\left[2bk + 2bk^R + kk^R + 3b^2 \right] > 0 \\
\Leftrightarrow \quad & [c_N - c_A]\left[2bk + 2bk^R + kk^R + 3b^2 \right] + k_N[(a - c_A)(2b + k) - b(a - c)] > 0 .
\end{aligned}$$

The inequality here holds because $c_N \geq c_A$, by assumption and because (3) implies:

$$k_A\,[\,(a-c_N)\,(2\,b+k)-b\,(a-c)\,]\;>\;[\,c_N-c_A\,]\left[\,2\,b\,k+2\,b\,k^R+k\,k^R+3\,b^2\,\right]$$

$$\Rightarrow\quad [\,a-c_N\,]\,[\,2\,b+k\,]-b\,[\,a-c\,]\;>\;0\quad\Rightarrow\quad [\,a-c_A\,]\,[\,2\,b+k\,]-b\,[\,a-c\,]\;>\;0. \tag{107}$$

The last two inequalities in (107) hold because $c_N \geq c_A$, by assumption. □ ■

Proposition 2. *In equilibrium, for* $\bar{p} \in (\bar{p}_2, \bar{p}_3)$, $\frac{dq_A}{d\overline{p}} < 0$, $\frac{dq_N}{d\overline{p}} < 0$, $\frac{dq}{d\overline{p}} > 0$, $\frac{dQ}{d\overline{p}} < 0$, *and* $\frac{dP(Q)}{d\overline{p}} = 1$.

Proof. (57) implies:

$$\frac{dq_A}{d\overline{p}} \;=\; -\,\frac{k_N\,[\,b+k\,]+b\,k_N}{b\,[\,b+k\,]\,[\,k_N+k_A\,]}\;<\;0\,;$$

$$\frac{dq_N}{d\overline{p}} \;=\; -\,\frac{k_A\,[\,b+k\,]+b\,k_A}{b\,[\,b+k\,]\,[\,k_N+k_A\,]}\;<\;0\,;\quad \frac{dq}{d\overline{p}} \;=\; \frac{1}{b+k}\;>\;0\,;$$

$$\frac{dQ}{d\overline{p}} \;=\; -\,\frac{1}{b}\;<\;0\quad\Rightarrow\quad \frac{dP(Q)}{d\overline{p}} \;=\; -\,b\left[-\,\frac{1}{b}\right]\;=\;1\,.\quad ■$$

Proposition 3. *For* $\bar{p} \in (\bar{p}_2, \bar{p}_3)$*: (i)* $V(\overline{p})$ *is a strictly concave function of* $\overline{p}$*; (ii)* $\frac{\partial V(\overline{p})}{\partial \overline{p}} \lesseqgtr 0 \Leftrightarrow \overline{p} \gtreqless \overline{p}_{V_3M}$ *where* $\overline{p}_{V_3M} \in [\,\overline{p}_2, \overline{p}_3)$*; and (iii)* $\overline{p}_{V_3M} = \overline{p}_2$ *if* $\Phi_1 \geq 0$*, whereas* $\overline{p}_{V_3M} > \overline{p}_2$ *if* $\Phi_1 < 0$*, where*

$$\Phi_1 \;\equiv\; \left[k^R+\frac{b^2}{2\,b+k}\right][\,k_A+k_N\,]\,A+2\,b\,[\,b+k\,]\,c_A\,[\,k_N+b\,]$$
$$+\;[\,2\,b\,(b+k)\,c_N+A\,k_N\,]\,[\,k_A-b\,]\quad \textit{where}\quad A\;\equiv\;a\,[\,b+k\,]+b\,c\,. \tag{108}$$

Proof. (62) implies that for $\bar{p} \in (\bar{p}_2, \bar{p}_3)$, R's revenue is:

$$V(\overline{p}) \;=\; \overline{p}\left[\frac{a\,(b+k)+b\,c-\overline{p}\;(2\,b+k)}{b\,[\,b+k\,]}\right] \;=\; \frac{[\,a\,(b+k)+b\,c\,]\,\overline{p}-[\,2\,b+k\,]\,\overline{p}^2}{b\,[\,b+k\,]}\,. \tag{109}$$

The value of $\bar{p}$ at which $V(\overline{p})$ in (109) is maximized is determined by:

$$a\,[\,b+k\,]+b\,c-2\,[\,2\,b+k\,]\,\overline{p}\;=\;0\quad\Rightarrow\quad\bar{p}\;=\;\frac{a\,[\,b+k\,]+b\,c}{2\,[\,2\,b+k\,]}\;\equiv\;\bar{p}_{V_3M}\,. \tag{110}$$

From (8):

$$\bar{p}_3 \;=\; \frac{[\,a\,(b+k)+b\,c\,]\left[\,\left(b+k^R\right)(k_N+k_A)+k_N\,k_A\,\right]+b\,c_N\,[\,b+k\,]\,k_A+b\,k_N\,[\,b+k\,]\,c_A}{b\,[\,b+k\,]\,[\,k_N+k_A\,]+k_N\,k_A\,[\,2\,b+k\,]+[\,k_N+k_A\,]\,[\,2\,b+k\,]\,[\,b+k^R\,]}$$

$$= \frac{\left[\left(b+k^R\right)(k_N+k_A)+k_N\,k_A\right]\frac{a[b+k]+bc}{b[b+k]}+c_N\;k_A+k_N\,c_A}{k_N+k_A+[(b+k^R)(k_N+k_A)+k_N\,k_A]\frac{2b+k}{b[b+k]}}. \tag{111}$$

(110) and (111) imply that $\bar{p}_{V_3M} < \bar{p}_3$ if:

$$\frac{a[b+k]+bc}{2[2b+k]} < \frac{\left[\left(b+k^R\right)(k_N+k_A)+k_N\,k_A\right]\frac{a[b+k]+bc}{b[b+k]}+c_N\;k_A+k_N\,c_A}{k_N+k_A+[(b+k^R)(k_N+k_A)+k_N\,k_A]\frac{2b+k}{b[b+k]}}. \tag{112}$$

The inequality in (112) holds if:

$$\frac{\left[\left(b+k^R\right)(k_N+k_A)+k_N\,k_A\right]\frac{a[b+k]+bc}{b[b+k]}}{k_N+k_A+[(b+k^R)(k_N+k_A)+k_N\,k_A]\frac{2b+k}{b[b+k]}} > \frac{a[b+k]+bc}{2[2b+k]}. \tag{113}$$

Define $z \equiv \left[\left(b+k^R\right)(k_N+k_A)+k_N\,k_A\right]\frac{1}{b[b+k]}$. Then the inequality in (113) holds if:

$$\frac{z[a(b+k)+bc]}{k_N+k_A+z[2b+k]} > \frac{a[b+k]+bc}{2[2b+k]}$$

$$\Leftrightarrow \quad \frac{z}{k_N+k_A+z[2b+k]} > \frac{1}{2[2b+k]}$$

$$\Leftrightarrow \quad 2z[2b+k] > k_N+k_A+z[2b+k] \quad\Leftrightarrow\quad [2b+k]z > k_N+k_A$$

$$\Leftrightarrow \quad \left[\left(b+k^R\right)(k_N+k_A)+k_N\,k_A\right]\frac{2b+k}{b[b+k]} > k_N+k_A$$

$$\Leftrightarrow \quad \frac{[2b+k]\left[b+k^R\right]}{b[b+k]}[k_N+k_A]+k_N\,k_A\left[\frac{2b+k}{b(b+k)}\right] > k_N+k_A$$

$$\Leftrightarrow \quad \frac{[2b+k]\left[b+k^R\right]-b[b+k]}{b[b+k]}[k_N+k_A]+k_N\,k_A\left[\frac{2b+k}{b(b+k)}\right] > 0. \tag{114}$$

The inequality in (114) always holds because:

$$\begin{aligned}[2b+k]\left[b+k^R\right]-b[b+k] &= 2b^2+2b\,k^R+b\,k+k\,k^R-b^2-b\,k\\ &= b^2+2b\,k^R+k\,k^R > 0.\end{aligned}$$

(114) implies that $\bar{p}_{V_3M} < \bar{p}_3$.

(109) and (110) imply that for $\bar{p} \in (\bar{p}_2, \bar{p}_3)$, $V(\bar{p})$ is a strictly concave function that attains its maximum at $\bar{p}_{V_3M}$. Therefore, $\frac{\partial V(\bar{p})}{\partial \bar{p}} < 0$ for $\bar{p} \in (\bar{p}_{V_3M}, \bar{p}_3)$.

(7) and (110) imply that $\bar{p}_2 \geq \bar{p}_{V_3M}$ if and only if:

$$\frac{1}{b[b+k][k_N+k_A]+[k_A\,k_N-k_N\,b][2b+k]+[k_N+k_A][2b+k][b+k^R]}$$

$$\cdot \left\{[(b+k)a+bc]\left[\left(b+k^R\right)(k_N+k_A)+k_N\,k_A-b\,k_N\right]\right.$$

$$+\ b\,[\,b+k\,]\,[\,k_A-b\,]\,c_N+b\,[\,k_N+b\,]\,[\,b+k\,]\,c_A\,\}$$

$$\geq\ \frac{a\,[\,b+k\,]+b\,c}{2\,[\,b+k\,]}$$

$$\Leftrightarrow\ \frac{1}{\frac{b[\,b+k\,]}{2\,b+k}\,[\,k_N+k_A\,]+[\,k_A\,k_N-k_N\,b\,]+[\,k_N+k_A\,]\,[\,b+k^R\,]}$$

$$\cdot\,\{\,2\,[\,(b+k)\,a+b\,c\,]\,\Big[\,\Big(b+k^R\Big)\,(k_N+k_A)+k_N\,k_A-b\,k_N\,\Big]$$

$$+\ 2\,b\,[\,b+k\,]\,[\,k_A-b\,]\,c_N+2\,b\,[\,k_N+b\,]\,[\,b+k\,]\,c_A\,\}$$

$$\geq\ a\,[\,b+k\,]+b\,c$$

$$\Leftrightarrow\ 2\,[\,(b+k)\,a+b\,c\,]\,\Big[\,\Big(b+k^R\Big)\,(k_N+k_A)+k_N\,k_A-b\,k_N\,\Big]+2\,b\,[\,b+k\,]\,[\,k_A-b\,]\,c_N$$

$$+\ 2\,b\,[\,k_N+b\,]\,[\,b+k\,]\,c_A$$

$$\geq\ [\,(b+k)\,a+b\,c\,]\,\frac{b\,[\,b+k\,]}{2\,b+k}\,[\,k_N+k_A\,]+[\,a\,(b+k)+b\,c\,]\,[\,k_A\,k_N-b\,k_N\,]$$

$$+\ [\,a\,(b+k)+b\,c\,]\,[\,k_N+k_A\,]\,\Big[\,b+k^R\,\Big]$$

$$\Leftrightarrow\ [\,(b+k)\,a+b\,c\,]\,\Big[\,\Big(b+k^R\Big)\,(k_N+k_A)+k_N\,k_A-b\,k_N\,\Big]+2\,b\,[\,b+k\,]\,[\,k_A-b\,]\,c_N$$

$$+\ 2\,b\,[\,k_N+b\,]\,[\,b+k\,]\,c_A\ \geq\ [\,a\,(b+k)+b\,c\,]\,\frac{b\,[\,b+k\,]}{2\,b+k}\,[\,k_N+k_A\,]$$

$$\Leftrightarrow\ [\,(b+k)\,a+b\,c\,]\left[\left(b-\frac{b\,(b+k)}{2\,b+k}+k^R\right)(k_N+k_A)+k_N\,k_A-b\,k_N\right]$$

$$+\ 2\,b\,[\,b+k\,]\,[\,k_A-b\,]\,c_N+2\,b\,[\,k_N+b\,]\,[\,b+k\,]\,c_A\ \geq\ 0$$

$$\Leftrightarrow\ [\,(b+k)\,a+b\,c\,]\left[\left(\frac{2\,b^2+k\,b}{2\,b+k}-\frac{b\,(b+k)}{2\,b+k}+k^R\right)(k_N+k_A)+k_N\,k_A-b\,k_N\right]$$

$$+\ 2\,b\,[\,b+k\,]\,[\,k_A-b\,]\,c_N+2\,b\,[\,k_N+b\,]\,[\,b+k\,]\,c_A\ \geq\ 0$$

$$\Leftrightarrow\ [\,(b+k)\,a+b\,c\,]\left[\left(\frac{b^2}{2\,b+k}+k^R\right)(k_N+k_A)+k_N\,k_A-b\,k_N\right]$$

$$+\ 2\,b\,[\,b+k\,]\,[\,k_A-b\,]\,c_N+2\,b\,[\,k_N+b\,]\,[\,b+k\,]\,c_A\ \geq\ 0$$

$$\Leftrightarrow\ [\,(b+k)\,a+b\,c\,]\left[\left(\frac{b^2}{2\,b+k}+k^R\right)(k_N+k_A)+k_N\,k_A-b\,k_N\right]$$

$$+\ 2\,b\,[\,b+k\,]\,[\,k_A\,c_N+k_N\,c_A-b\,(c_N-c_A)\,]\ \geq\ 0$$

$$\Leftrightarrow \quad \left[\frac{(b+k)\,a + b\,c}{2\,b+k} \right] \left[\left(b^2 + k^R\,[\,2\,b+k\,]\right)(k_N + k_A) + k_N\,k_A\,(2\,b+k) - b\,k_N\,(2\,b+k) \,\right]$$

$$+\ 2\,b\,[\,b+k\,]\,[\,k_A\,c_N + k_N\,c_A - b\,(c_N - c_A)\,] \ \geq\ 0\,. \tag{115}$$

Observe that:

$$\left[\,b^2 + k^R\,(2\,b+k)\,\right][\,k_N + k_A\,] + k_N\,k_A\,[\,2\,b+k\,] - b\,k_N\,[\,2\,b+k\,]$$

$$= \ b^2\,[\,k_N + k_A\,] + k^R\,[\,2\,b+k\,]\,[\,k_N + k_A\,] + k_N\,[\,2\,b+k\,]\,[\,k_A - b\,]$$

$$= \ b^2\,[\,k_N + k_A\,] - b\,[\,2\,b+k\,]\,[\,k_N + k_A\,] + b\,[\,2\,b+k\,]\,[\,k_N + k_A\,]$$

$$+\ \ k^R\,[\,2\,b+k\,]\,[\,k_N + k_A\,] + k_N\,[\,2\,b+k\,]\,[\,k_A - b\,]$$

$$= \ -\,b\,[\,b+k\,]\,[\,k_N + k_A\,] + \left[\,b + k^R\,\right][\,2\,b+k\,]\,[\,k_N + k_A\,] + k_N\,[\,2\,b+k\,]\,[\,k_A - b\,]$$

$$= \ -\,2\,b\,[\,b+k\,]\,[\,k_N + k_A\,] + D_2\,. \tag{116}$$

The last equality in (116) reflects (7). (115) and (116) imply:

$$\bar{p}_2 \ \geq\ \bar{p}_{V_3M} \quad \Leftrightarrow \quad \widetilde{\Phi}_1 \ \geq\ 0\,,$$

$$\text{where} \ \ \widetilde{\Phi}_1 \ \equiv\ \left[\frac{(b+k)\,a + b\,c}{2\,b+k} \right] \{\, D_2 - 2\,b\,[\,b+k\,]\,[\,k_N + k_A\,]\,\}$$

$$+\ \ 2\,b\,[\,b+k\,]\,[\,k_A\,c_N + k_N\,c_A - b\,(c_N - c_A)\,]\,. \tag{117}$$

(7) implies:

$$D_2 - 2\,b\,[\,b+k\,]\,[\,k_N + k_A\,] \ = \ -\,b\,[\,b+k\,]\,[\,k_N + k_A\,] + k_N\,[\,k_A - b\,]\,[\,2\,b+k\,]$$

$$+\ [\,2\,b+k\,]\left[\,b + k^R\,\right][\,k_N + k_A\,]$$

$$= \ [\,k_A + k_N\,]\left[\,2\,b^2 + 2\,b\,k^R + b\,k + k\,k^R - b^2 - b\,k\,\right] + k_N\,[\,k_A - b\,]\,[\,2\,b+k\,]$$

$$= \ [\,k_A + k_N\,]\left[\,b^2 + 2\,b\,k^R + k\,k^R\,\right] + k_N\,[\,k_A - b\,]\,[\,2\,b+k\,]\,. \tag{118}$$

(118) implies:

$$\left[\frac{(b+k)\,a + b\,c}{2\,b+k} \right] [\,D_2 - 2\,b\,(b+k)\,(k_N + k_A)\,]$$

$$= \ \left[\frac{(b+k)\,a + b\,c}{2\,b+k} \right] [\,k_A + k_N\,]\left[\,b^2 + k^R\,(2\,b+k)\,\right] + [\,(b+k)\,a + b\,c\,]\,k_N\,[\,k_A - b\,]\,. \tag{119}$$

(108) and (119) imply:

$$\widetilde{\Phi}_1 \ = \ \frac{b^2}{2\,b+k}\,[\,k_A + k_N\,]\,[\,(b+k)\,a + b\,c\,] + k^R\,[\,k_A + k_N\,]\,[\,(b+k)\,a + b\,c\,]$$

$$+ [\,(b+k)\,a + b\,c\,]\,k_N\,[\,k_A - b\,] + 2\,b\,[\,b+k\,]\,c_N\,[\,k_A - b\,]$$

$$+ 2\,b\,[\,b+k\,]\,c_A\,[\,k_N + b\,]$$

$$= \left[k^R + \frac{b^2}{2\,b+k} \right] [\,k_A + k_N\,]\,[\,(b+k)\,a + b\,c\,] + 2\,b\,[\,b+k\,]\,c_A\,[\,k_N + b\,]$$

$$+ \{\,2\,b\,[\,b+k\,]\,c_N + [\,(b+k)\,a + b\,c\,]\,k_N\,\}\,[\,k_A - b\,] \;\equiv\; \Phi_1\,. \quad \blacksquare$$

Proposition 4. *$\bar{p}_3 - \bar{p}_2$ increases as: (i) c_A, k_A, or k^R declines; (ii) c or c_N increases; or (iii) k_N increases if $k_A - b$ is sufficiently small.*

Proof. (7) and (8) imply:

$$\bar{p}_2 \;=\; \frac{N_2}{D_2} \quad \text{and} \quad \bar{p}_3 \;=\; \frac{N_3}{D_2 + b\,[\,2\,b+k\,]\,k_N}$$

$$\text{where} \quad N_3 \;\equiv\; [\,a\,(b+k) + b\,c\,]\,\left[\,\left(b + k^R\right)(k_N + k_A) + k_N\,k_A\,\right]$$

$$+ \,b\,c_N\,[\,b+k\,]\,k_A + b\,k_N\,[\,b+k\,]\,c_A \quad \text{and}$$

$$N_2 \;\equiv\; [\,a\,(b+k) + b\,c\,]\,\left[\,\left(b + k^R\right)(k_N + k_A) + k_N\,k_A\,\right]$$

$$+ \,b\,c_N\,[\,b+k\,]\,k_A + b\,k_N\,[\,b+k\,]\,c_A$$

$$- \;b\,k_N\,[\,a\,(b+k) + b\,c\,] - b^2\,[\,b+k\,]\,c_N + b^2\,[\,b+k\,]\,c_A$$

$$= \; N_3 - b\,k_N\,[\,a\,(b+k) + b\,c\,] - b^2\,[\,b+k\,]\,[\,c_N - c_A\,]\,. \tag{120}$$

To prove that $\frac{\partial(\bar{p}_3 - \bar{p}_2)}{\partial k^R} < 0$, let $q_A(\bar{p})$ denote R's equilibrium output using A's input when the price cap is $\bar{p} \in [\,\bar{p}_2, \bar{p}_3\,]$. Let $q_N(\bar{p})$ denote R's corresponding output when R does not employ A's input. Also let $Q^R(\bar{p}) = q_A(\bar{p}) + q_N(\bar{p})$. (73) implies:

$$\bar{p}_3 \;=\; \left[\,b + k^R\,\right] Q^R(\bar{p}_3) + c_N + k_N\,q_N(\bar{p}_3)$$

where, from (57):

$$q_N(\bar{p}_3) \;=\; \frac{k_A\,[\,b+k\,]\,[\,a - \bar{p}_3\,] - b\,k_A\,[\,\bar{p}_3 - c\,] - b\,[\,b+k\,]\,[\,c_N - c_A\,]}{b\,[\,b+k\,]\,[\,k_N + k_A\,]} \quad \text{and}$$

$$Q^R(\bar{p}_3) \;=\; \frac{[\,b+k\,]\,[\,a - \bar{p}_3\,] - b\,[\,\bar{p}_3 - c\,]}{b\,[\,b+k\,]}\,. \tag{121}$$

(121) implies that $q_N(\bar{p}_3)$ and $Q^R(\bar{p}_3)$ vary with k^R only through $\bar{p}_3$. Therefore, (121) implies:

$$\frac{\partial \bar{p}_3}{\partial k^R} \;=\; Q^R(\bar{p}_3) + \left[\,b + k^R\,\right] \frac{\partial Q^R(\bar{p}_3)}{\partial \bar{p}_3}\,\frac{\partial \bar{p}_3}{\partial k^R} + k_N\,\frac{\partial q_N(\bar{p}_3)}{\partial \bar{p}_3}\,\frac{\partial \bar{p}_3}{\partial k^R}\,. \tag{122}$$

(121) also implies:

$$\frac{\partial q_N(\bar{p}_3)}{\partial \bar{p}_3} \;=\; -\,\frac{k_A\,[\,b+k\,]+b\,k_A}{b\,[\,b+k\,]\,[\,k_N+k_A\,]} \;\equiv\; D_N \;<\; 0\,;$$

$$\frac{\partial Q^R(\bar{p}_3)}{\partial \bar{p}_3} \;=\; -\,\frac{2\,b+k}{b\,[\,b+k\,]} \;\equiv\; D_R \;<\; 0\,. \tag{123}$$

(122) and (123) imply:

$$\frac{\partial \bar{p}_3}{\partial k^R} \;=\; Q^R(\bar{p}_3)+\left[\,b+k^R\,\right] D_R\,\frac{\partial \bar{p}_3}{\partial k^R}+k_N\,D_N\,\frac{\partial \bar{p}_3}{\partial k^R}$$

$$\Rightarrow\quad \frac{\partial \bar{p}_3}{\partial k^R}\left[\,1-\left(b+k^R\right)D_R-k_N\,D_N\,\right] \;=\; Q^R(\bar{p}_3)$$

$$\Rightarrow\quad \frac{\partial \bar{p}_3}{\partial k^R} \;=\; \frac{Q^R(\bar{p}_3)}{1-[\,b+k^R\,]\,D_R-k_N\,D_N} \;>\; 0\,. \tag{124}$$

The inequality in (124) holds because $D_R < 0$ and $D_N < 0$, from (123).

(73) implies:

$$\bar{p}_2 \;=\; \left[\,b+k^R\,\right] Q^R(\bar{p}_2)+c_N+k_N\,q_N(\bar{p}_2)-b\,q_A(\bar{p}_2)$$

where, from (57):

$$q_A(\bar{p}_2) \;=\; \frac{b\,[\,b+k\,]\,[\,c_N-c_A\,]+k_N\,[\,a-\bar{p}\,]\,[\,b+k\,]-b\,k_N\,[\,\bar{p}-c\,]}{b\,[\,b+k\,]\,[\,k_N+k_A\,]}\,;$$

$$q_N(\bar{p}_2) \;=\; \frac{k_A\,[\,b+k\,]\,[\,a-\,\bar{p}_2\,]-b\,k_A\,[\,\bar{p}_2-c\,]-b\,[\,b+k\,]\,[\,c_N-c_A\,]}{b\,[\,b+k\,]\,[\,k_N+k_A\,]}\,;\;\text{ and}$$

$$Q^R(\bar{p}_2) \;=\; \frac{[\,b+k\,]\,[\,a-\,\bar{p}_2\,]-b\,[\,\bar{p}_2-c\,]}{b\,[\,b+k\,]}\,. \tag{125}$$

(125) implies that $q_A(\bar{p}_2)$, $q_N(\bar{p}_2)$, and $Q^R(\bar{p}_2)$ vary with k^R only through $\bar{p}_2$. Therefore, (125) implies:

$$\frac{\partial \bar{p}_2}{\partial k^R} \;=\; Q^R(\bar{p}_2)+\left[\,b+k^R\,\right]\frac{\partial Q^R(\bar{p}_2)}{\partial \bar{p}_2}\,\frac{\partial \bar{p}_2}{\partial k^R}$$

$$+\;\; k_N\,\frac{\partial q_N(\bar{p}_2)}{\partial \bar{p}_2}\,\frac{\partial \bar{p}_2}{\partial k^R}-b\,\frac{\partial q_A(\bar{p}_2)}{\partial \bar{p}_2}\,\frac{\partial \bar{p}_2}{\partial k^R}\,. \tag{126}$$

(125) also implies:

$$\frac{\partial q_A(\bar{p}_2)}{\partial \bar{p}_2} \;=\; -\,\frac{k_N\,[\,b+k\,]+b\,k_N}{b\,[\,b+k\,]\,[\,k_N+k_A\,]} \;\equiv\; D_A \;<\; 0\,;$$

$$\frac{\partial q_N(\bar{p}_2)}{\partial \bar{p}_2} \;=\; -\,\frac{k_A\,[\,b+k\,]+b\,k_A}{b\,[\,b+k\,]\,[\,k_N+k_A\,]} \;\equiv\; D_N \;<\; 0\,;$$

$$\frac{\partial Q^R(\bar{p}_2)}{\partial \bar{p}_2} \;=\; -\,\frac{2\,b+k}{b\,[\,b+k\,]} \;\equiv\; D_R \;<\; 0\,. \tag{127}$$

(126) and (127) imply:

$$\frac{\partial \bar{p}_2}{\partial k^R} \;=\; Q^R(\bar{p}_2)+\left[\,b+k^R\,\right]D_R\,\frac{\partial \bar{p}_2}{\partial k^R}+k_N\,D_N\,\frac{\partial \bar{p}_2}{\partial k^R}-b\,D_A\,\frac{\partial \bar{p}_2}{\partial k^R}$$

$$\Rightarrow\;\; \frac{\partial \bar{p}_2}{\partial k^R}\,\left[\,1-\left(\,b+k^R\,\right)D_R-k_N D_N+b\,D_A\,\right] \;=\; Q^R(\bar{p}_2)$$

$$\Rightarrow\;\; \frac{\partial \bar{p}_2}{\partial k^R} \;=\; \frac{Q^R(\bar{p}_2)}{1-[\,b+k^R\,]\,D_R-k_N\,D_N+b\,D_A}\,. \tag{128}$$

(127) implies:

$$-\,b\,D_R+b\,D_A \;=\; b\,[\,-\,D_R+D_A\,] \;=\; b\left[\,\frac{2\,b+k}{b\,(b+k)}-\frac{k_N\,(b+k)+b\,k_N}{b\,(b+k)\,(k_N+k_A)}\,\right]$$

$$=\; b\left[\,\frac{2\,b+k}{b\,(b+k)}-\left(\frac{k_N}{k_N+k_A}\right)\frac{2\,b+k}{b\,(b+k)}\,\right]$$

$$=\; b\left[\,\frac{2\,b+k}{b\,(b+k)}\,\right]\left[\,1-\frac{k_N}{k_N+k_A}\,\right] \;>\; 0 \tag{129}$$

Because $D_N \;<\; 0$ and $D_R \;<\; 0$ from (127), (129) implies:

$$1-\left[\,b+k^R\,\right]D_R-k_N\,D_N+b\,D_A \;=\; 1-k^R\,D_R-k_N\,D_N-b\,D_R+b\,D_A$$

$$>\; 1-k^R\,D_R-k_N\,D_N \;>\; 0\,. \tag{130}$$

Because $D_A < 0$ from (127), (130) implies:

$$1-\left[\,b+k^R\,\right]D_R-k_N\,D_N \;>\; 0\,. \tag{131}$$

(128) and (130) imply:

$$\frac{\partial \bar{p}_2}{\partial k^R} \;=\; \frac{Q^R(\bar{p}_2)}{1-[\,b+k^R\,]\,D_R-k_N\,D_N+b\,D_A} \;>\; 0\,. \tag{132}$$

(124) and (130) – (132) imply:

$$\frac{\partial \bar{p}_3}{\partial k^R}-\frac{\partial \bar{p}_2}{\partial k^R} \;=\; \frac{Q^R(\bar{p}_3)}{1-[\,b+k^R\,]\,D_R-k_N\,D_N}-\frac{Q^R(\bar{p}_2)}{1-[\,b+k^R\,]\,D_R-k_N\,D_N+b\,D_A} \;<\; 0$$

$$\Leftrightarrow \quad \frac{Q^R(\bar{p}_3)}{1 - [\,b + k^R\,]\,D_R - k_N\,D_N} \;<\; \frac{Q^R(\bar{p}_2)}{1 - [\,b + k^R\,]\,D_R - k_N\,D_N + b\,D_A}$$

$$\Leftrightarrow \quad \frac{Q^R(\bar{p}_3)}{Q^R(\bar{p}_2)} \;<\; \frac{1 - \left[\,b + k^R\,\right] D_R - k_N\,D_N}{1 - [\,b + k^R\,]\,D_R - k_N\,D_N + b\,D_A}\,. \tag{133}$$

(62) implies that $Q^R(\bar{p}_3) \,<\, Q^R(\bar{p}_2)$. Therefore:

$$\frac{Q^R(\bar{p}_3)}{Q^R(\bar{p}_2)} \;<\; 1\,. \tag{134}$$

Furthermore, because $1 - \left[\,b + k^R\,\right] D_R - k_N D_N + b D_A \,>\, 0$ from (130):

$$\frac{1 - \left[\,b + k^R\,\right] D_R - k_N\,D_N}{1 - [\,b + k^R\,]\,D_R - k_N\,D_N + b\,D_A} \;>\; 1$$

$$\Leftrightarrow \quad 1 - \left[\,b + k^R\,\right] D_R - k_N\,D_N \;>\; 1 - \left[\,b + k^R\,\right] D_R - k_N\,D_N + b\,D_A$$

$$\Leftrightarrow \quad D_A \;<\; 0\,. \tag{135}$$

(127) implies that the last inequality in (135) holds. (134) and (135) imply that (133) holds. Therefore, because $\bar{p}_3 \,>\, \bar{p}_2 \,>\, 0$ from Proposition 1, (124) and (133) imply that $\frac{\partial(\bar{p}_3 - \bar{p}_2)}{\partial k^R} < 0\,.$

To prove that $\frac{\partial(\,\bar{p}_3\,-\,\bar{p}_2)}{\partial c_N} \,>\, 0$, observe that (7) and (8) imply:

$$\frac{\partial \bar{p}_2}{\partial c_N} \;=\; \frac{b\,[\,b + k\,]\,[\,k_A - b\,]}{D_2} \quad \text{and} \quad \frac{\partial \bar{p}_3}{\partial c_N} \;=\; \frac{b\,[\,b + k\,]\,k_A}{D_2 + b\,k_N[\,2\,b + k\,]}\,. \tag{136}$$

(136) implies:

$$\frac{\partial \bar{p}_3}{\partial c_N} - \frac{\partial \bar{p}_2}{\partial c_N} \;=\; \frac{b\,[\,b + k\,]\,k_A}{D_2 + b\,k_N\,[\,2\,b + k\,]} - \frac{b\,[\,b + k\,]\,[\,k_A - b\,]}{D_2} \;>\; 0$$

$$\Leftrightarrow \quad \frac{k_A}{D_2 + b\,k_N\,[\,2\,b + k\,]} \;>\; \frac{k_A - b}{D_2}$$

$$\Leftrightarrow \quad D_2\,k_A \;>\; [\,D_2 + b\,k_N\,(2\,b + k)\,]\,[\,k_A - b\,]$$

$$\Leftrightarrow \quad D_2\,k_A \;>\; D_2\,k_A - b\,D_2 + b\,k_N\,[\,2\,b + k\,]\,[\,k_A - b\,]$$

$$\Leftrightarrow \quad D_2 - k_N\,[\,2\,b + k\,]\,[\,k_A - b\,] \;>\; 0\,. \tag{137}$$

The inequality in (137) holds because, from (7):

$$D_2 - k_N\,[\,2\,b + k\,]\,[\,k_A - b\,]$$

$$= \; b\,[\,b + k\,]\,[\,k_N + k_A\,] + [\,2\,b + k\,]\,k_N\,[\,k_A - b\,]$$

$$+ [\,2\,b + k\,]\,[\,k_N + k_A\,]\left[\,b + k^R\,\right] - k_N\,[\,2\,b + k\,]\,[\,k_A - b\,]$$

$$= \; b\,[\,b + k\,]\,[\,k_N + k_A\,] + [\,2\,b + k\,]\,[\,k_N + k_A\,]\left[\,b + k^R\,\right] \; > \; 0\,.$$

To prove that $\frac{\partial(\,\bar{p}_3 - \bar{p}_2)}{\partial c_A} < 0$, observe that (7) and (8) imply:

$$\frac{\partial \bar{p}_2}{\partial c_A} \; = \; \frac{b\,[\,b + k\,]\,[\,k_N + b\,]}{D_2} \quad \text{and} \quad \frac{\partial \bar{p}_3}{\partial c_A} \; = \; \frac{b\,[\,b + k\,]\,k_N}{D_2 + b\,k_N[\,2\,b + k\,]}\,. \tag{138}$$

(138) implies:

$$\frac{\partial \bar{p}_3}{\partial c_A} - \frac{\partial \bar{p}_2}{\partial c_A} \; = \; \frac{b\,[\,b + k\,]\,k_N}{D_2 + b\,k_N\,[\,2\,b + k\,]} - \frac{b\,[\,b + k\,]\,[\,k_N + b\,]}{D_2} \; < \; 0$$

$$\Leftrightarrow \quad \frac{b\,[\,b + k\,]\,k_N}{D_2 + b\,k_N\,[\,2\,b + k\,]} \; < \; \frac{b\,[\,b + k\,]\,[\,k_N + b\,]}{D_2}$$

$$\Leftrightarrow \quad b\,D_2\,[\,b + k\,]\,k_N \; < \; [\,D_2 + b\,k_N\,(2\,b + k)\,]\,b\,[\,b + k\,]\,[\,k_N + b\,]$$

$$\Leftrightarrow \quad D_2\,k_N \; < \; [\,D_2 + b\,k_N\,(2\,b + k)\,]\,[\,k_N + b\,]$$

$$\Leftrightarrow \quad D_2\,k_N \; < \; D_2\,k_N + b\,D_2 + b\,k_N\,[\,2\,b + k\,]\,[\,k_N + b\,]$$

$$\Leftrightarrow \quad D_2 + k_N\,[\,2\,b + k\,]\,[\,k_N + b\,] \; > \; 0\,. \tag{139}$$

The inequality in (139) holds because, from (7):

$$D_2 \; = \; b\,[\,b + k\,]\,[\,k_N + k_A\,] + [\,2\,b + k\,]\left\{\,k_N\,[\,k_A - b\,] + [\,k_N + k_A\,]\left[\,b + k^R\,\right]\,\right\}$$

$$= \; b\,[\,b + k\,]\,[\,k_N + k_A\,] + [\,2\,b + k\,]\left\{\,k_N\left[\,k_A + k^R\,\right] + k_A\,\left[\,b + k^R\,\right]\,\right\} \; > \; 0\,.$$

To prove that $\frac{\partial(\bar{p}_3 - \bar{p}_2)}{\partial k_A} < 0$, we introduce the following:

<u>Definition</u>. $Y_1 \; \equiv \; b\,[\,b + k\,]\,\{\,k_N\,[\,a\,(b + k) + b\,c - (2\,b + k)\,c_A\,]$

$$+ \; [\,c_N - c_A\,]\left[\,b\,(b + k) + (2\,b + k)\left(b + k^R\right)\,\right]\,\}\,. \tag{140}$$

Observe that:

$$Y_1 \; > \; 0\,. \tag{141}$$

(141) holds because $c_N \; \geq \; c_A$ by assumption, and (3) implies:

$$[\,a - c_A\,]\,[\,2\,b + k\,] - b\,[\,a - c\,] \; > \; 0 \quad \Rightarrow \quad a\,[\,b + k\,] + b\,c - [\,2\,b + k\,]\,c_A \; > \; 0\,.$$

(8) implies:

$$(D_3)^2\,\frac{\partial \bar{p}_3}{\partial k_A} \; = \; D_3\left\{\,[\,a\,(b + k) + b\,c\,]\left[\,b + k^R + k_N\,\right] + b\,[\,b + k\,]\,c_N\,\right\}$$

$$- \Big\{ b\,[\,b+k\,] + [\,2\,b+k\,]\,\Big[\,k_N + b + k^R\,\Big]\,\Big\}$$

$$\cdot\;\{\,[\,a\,(b+k) + b\,c\,]\,\Big[\,\Big(b+k^R\Big)\,(k_A + k_N) + k_A\,k_N\,\Big]$$

$$+\;\; b\;[\,b+k\,]\,[\,c_N\,k_A + c_A\,k_N\,]\,\}$$

$$= \;\Big\{\,[\,a\,(b+k) + b\,c\,]\,\Big[\,b + k^R + k_N\,\Big] + b\,[\,b+k\,]\,c_N\,\Big\}$$

$$\cdot\,\Big\{\,b\,[\,b+k\,]\,[\,k_A + k_N\,] + [\,2\,b+k\,]\,\Big[\,k_A\,k_N + (k_A + k_N)\,\Big(b+k^R\Big)\,\Big]\,\Big\}$$

$$- \Big\{\,b\,[\,b+k\,] + [\,2\,b+k\,]\,\Big[\,k_N + b + k^R\,\Big]\,\Big\}$$

$$\cdot\,\{\,[\,a\,(b+k) + b\,c\,]\,\Big[\,\Big(b+k^R\Big)\,(k_A + k_N) + k_A\,k_N\,\Big]$$

$$+\;\; b\;[\,b+k\,]\,[\,c_N\,k_A + c_A\,k_N\,]\,\}$$

$$= \;[\,a\,(b+k) + b\,c\,]\,\Big[\,b + k^R + k_N\,\Big]\,b\,[\,b+k\,]\,[\,k_A + k_N\,]$$

$$+\;[\,a\,(b+k) + b\,c\,]\,\Big[\,b + k^R + k_N\,\Big]\,[\,2\,b+k\,]\,\Big[\,\Big(b+k^R\Big)\,(k_A + k_N) + k_A\,k_N\,\Big]$$

$$+\;b\,[\,b+k\,]\,c_N\,b\,[\,b+k\,]\,[\,k_A + k_N\,]$$

$$+\;b\,[\,b+k\,]\,c_N\,[\,2\,b+k\,]\,\Big[\,\Big(b+k^R\Big)\,(k_A + k_N) + k_A\,k_N\,\Big]$$

$$-\;b\,[\,b+k\,]\,[\,a\,(b+k) + b\,c\,]\,\Big[\,\Big(b+k^R\Big)\,(k_A + k_N) + k_A\,k_N\,\Big]$$

$$-\;b\,[\,b+k\,]\,b\;[\,b+k\,]\,[\,c_N\,k_A + c_A\,k_N\,]$$

$$-\;[\,2\,b+k\,]\,\Big[\,k_N + b + k^R\,\Big]\,[\,a\,(b+k) + b\,c\,]\,\Big[\,\Big(b+k^R\Big)\,(k_A + k_N) + k_A\,k_N\,\Big]$$

$$-\;[\,2\,b+k\,]\,\Big[\,k_N + b + k^R\,\Big]\,b\;[\,b+k\,]\,[\,c_N\,k_A + c_A\,k_N\,]\;\equiv\;\Phi\,. \tag{142}$$

(142) implies:

$$\Phi \;=\; [\,a\,(b+k) + b\,c\,]\,\Phi_A + b\,[\,b+k\,]\,\Phi_B \tag{143}$$

where

$$\Phi_A \;\equiv\; b\,[\,b+k\,]\,\Big[\,b + k^R + k_N\,\Big]\,[\,k_A + k_N\,]$$

$$+\;[\,2\,b+k\,]\,\Big[\,b + k^R + k_N\,\Big]\,\Big[\,\Big(b+k^R\Big)\,(k_A + k_N) + k_A\,k_N\,\Big]$$

$$-\;b\,[\,b+k\,]\,\Big[\,\Big(b+k^R\Big)\,(k_A + k_N) + k_A\,k_N\,\Big]$$

$$-\;[\,2\,b+k\,]\,\Big[\,b + k^R + k_N\,\Big]\,\Big[\,\Big(b+k^R\Big)\,(k_A + k_N) + k_A\,k_N\,\Big]$$

$$=\;b\,[\,b+k\,]\,\Big\{\,\Big[\,b + k^R + k_N\,\Big]\,[\,k_A + k_N\,] - \Big[\,\Big(b+k^R\Big)\,(k_A + k_N) + k_A\,k_N\,\Big]\,\Big\}$$

$$=\;b\,[\,b+k\,]\,\{\,k_N\,[\,k_A + k_N\,] - k_A\,k_N\,\}\;=\;b\,[\,b+k\,]\,(k_N)^2 \quad \text{and} \tag{144}$$

$$\Phi_B \equiv c_N\, b\,[\,b+k\,]\,[\,k_A+k_N\,] + c_N\,[\,2\,b+k\,]\left[\,\left(b+k^R\right)(k_A+k_N)+k_A\,k_N\,\right]$$

$$-\; b\,[\,b+k\,]\,[\,c_N\,k_A+c_A\,k_N\,] - [\,2\,b+k\,]\left[\,k_N+b+k^R\,\right][\,c_N\,k_A+c_A\,k_N\,]$$

$$= b\,[\,b+k\,]\,k_N\,[\,c_N-c_A\,] + [\,2\,b+k\,]\,\Phi_C \tag{145}$$

where

$$\Phi_C \equiv c_N\left[\,\left(b+k^R\right)(k_A+k_N)+k_A\,k_N\,\right] - \left[\,k_N+b+k^R\,\right][\,c_N\,k_A+c_A\,k_N\,]$$

$$= c_N\left[\,\left(b+k^R\right)(k_A+k_N)+k_A\,k_N-k_A\left(k_N+b+k^R\right)\,\right] - c_A\,k_N\left[\,k_N+b+k^R\,\right]$$

$$= c_N\,k_N\left[\,b+k^R\,\right] - c_A\,k_N\left[\,k_N+b+k^R\,\right] = \left[\,b+k^R\,\right]k_N\,[\,c_N-c_A\,] - c_A\,(k_N)^2\,. \tag{146}$$

(145) and (146) imply:

$$\Phi_B = b\,[\,b+k\,]\,k_N\,[\,c_N-c_A\,] + [\,2\,b+k\,]\left\{\,\left[\,b+k^R\,\right]k_N\,[\,c_N-c_A\,] - c_A\,(k_N)^2\,\right\}$$

$$= k_N\,[\,c_N-c_A\,]\left\{\,b\,[\,b+k\,] + [\,2\,b+k\,]\left[\,b+k^R\,\right]\,\right\} - [\,2\,b+k\,]\,c_A\,(k_N)^2\,. \tag{147}$$

(140), (143), (144), and (147) imply:

$$\Phi = [\,a\,(b+k)+b\,c\,]\,b\,[\,b+k\,]\,(k_N)^2$$

$$+\, b\,[\,b+k\,]\,k_N\,[\,c_N-c_A\,]\left\{\,b\,[\,b+k\,] + [\,2\,b+k\,]\left[\,b+k^R\,\right]\,\right\}$$

$$-\, b\,[\,b+k\,]\,[\,2\,b+k\,]\,c_A\,(k_N)^2$$

$$= b\,[\,b+k\,]\,k_N\,\{\,[\,a\,(b+k)+b\,c\,]\,k_N - [\,2\,b+k\,]\,c_A\,k_N$$

$$+\; [\,c_N-c_A\,]\left[\,b\,(b+k) + (2\,b+k)\,\left(b+k^R\right)\,\right]\,\}$$

$$= b\,[\,b+k\,]\,k_N\,\{\,k_N\,[\,a\,(b+k)+b\,c-(2\,b+k)\,c_A\,]$$

$$+\; [\,c_N-c_A\,]\left[\,b\,(b+k) + (2\,b+k)\,\left(b+k^R\right)\,\right]\,\} = k_N\,Y_1\,. \tag{148}$$

(141), (142), and (148) imply that $\frac{\partial \bar{p}_3}{\partial k_A} = \frac{k_N\,Y_1}{(D_3)^2} > 0$.

(7) implies:

$$(D_2)^2\,\frac{\partial \bar{p}_2}{\partial k_A} = D_2\left\{\,[\,a\,(b+k)+b\,c\,]\left[\,b+k^R+k_N\,\right] + b\,[\,b+k\,]\,c_N\,\right\}$$

$$-\left\{\,b\,[\,b+k\,] + [\,2\,b+k\,]\left[\,k_N+b+k^R\,\right]\,\right\}$$

$$\cdot\;\{\,[\,a\,(b+k)+b\,c\,]\left[\,\left(b+k^R\right)(k_A+k_N)+k_A\,k_N-b\,k_N\,\right]$$

$$+\; b\,[\,b+k\,]\,[\,c_N\,(k_A-b)+c_A\,(k_N+b)\,]\,\}$$

$$
\begin{aligned}
&= \Big\{ \left[a\,(b+k) + b\,c \right] \left[b + k^R + k_N \right] + b\left[b+k \right] c_N \Big\} \\
&\qquad \cdot \Big\{ b\left[b+k \right]\left[k_A + k_N \right] + \left[2\,b + k \right] \left[k_N \left(k_A + k^R \right) + k_A \left(b + k^R \right) \right] \Big\} \\
&\quad - \Big\{ b\left[b+k \right] + \left[2\,b+k \right] \left[k_N + b + k^R \right] \Big\} \\
&\qquad\quad \cdot \Big\{ \left[a\,(b+k) + b\,c \right] \left[k_A\,(b+k_N) + k^R\,(k_A+k_N) \right] \\
&\qquad\qquad\qquad + b\left[b+k \right]\left[c_N\,(k_A - b) + c_A\,(k_N+b) \right] \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \left[a\,(b+k) + b\,c \right] \left[b + k^R + k_N \right] b\left[b+k \right]\left[k_N + k_A \right] \\
&\quad + \left[a\,(b+k) + b\,c \right] \left[b + k^R + k_N \right] \left[2\,b+k \right] \left[k_N \left(k_A + k^R \right) + k_A \left(b + k^R \right) \right] \\
&\quad + b\left[b+k \right] c_N\, b\left[b+k \right]\left[k_N + k_A \right] \\
&\quad + b\left[b+k \right] c_N \left[2\,b+k \right] \left[k_N \left(k_A + k^R \right) + k_A \left(b + k^R \right) \right] \\
&\quad - b\left[b+k \right]\left[a\,(b+k) + b\,c \right] \left[k_A\,(b+k_N) + k^R\,(k_A+k_N) \right] \\
&\quad - b\left[b+k \right] b\left[b+k \right]\left[c_N\,(k_A - b) + c_A\,(k_N+b) \right] \\
&\quad - \left[2\,b+k \right] \left[k_N + b + k^R \right] \left[a\,(b+k) + b\,c \right] \left[k_A\,(b+k_N) + k^R\,(k_A+k_N) \right] \\
&\quad - \left[2\,b+k \right] \left[k_N + b + k^R \right] b\left[b+k \right]\left[c_N\,(k_A - b) + c_A\,(k_N+b) \right] \;\equiv\; \mathcal{F}\,. \qquad (149)
\end{aligned}
$$

(149) implies:

$$
\mathcal{F} \;=\; \left[a\,(b+k) + b\,c \right] \mathcal{F}_1 + b\left[b+k \right] \mathcal{F}_2 \qquad (150)
$$

where

$$
\begin{aligned}
\mathcal{F}_1 \;\equiv\;& \left[b + k^R + k_N \right] b\left[b+k \right]\left[k_N + k_A \right] \\
&+ \left[2\,b+k \right] \left[b + k^R + k_N \right] \left[k_N \left(k_A + k^R \right) + k_A \left(b + k^R \right) \right] \\
&- b\left[b+k \right] \left[k_A\,(b+k_N) + k^R\,(k_A+k_N) \right] \\
&- \left[2\,b+k \right] \left[k_N + b + k^R \right] \left[k_A\,(b+k_N) + k^R\,(k_A+k_N) \right] \\
=\;& b\left[b+k \right] \Big\{ \left[b + k^R + k_N \right] \left[k_N + k_A \right] - \left[k_A\,(b+k_N) + k^R\,(k_A+k_N) \right] \Big\} \\
=\;& b\left[b+k \right] \left\{ \left[b + k_N \right] \left[k_N + k_A \right] - k_A \left[b + k_N \right] \right\} \;=\; b\left[b+k \right] \left[b + k_N \right] k_N \qquad (151)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{F}_2 \;\equiv\;& b\left[b+k \right]\left[k_N + k_A \right] c_N + c_N \left[2\,b+k \right] \left[k_N \left(k_A + k^R \right) + k_A \left(b + k^R \right) \right] \\
&- b\left[b+k \right]\left[c_N\,(k_A - b) + c_A\,(k_N+b) \right]
\end{aligned}
$$

$$
\begin{aligned}
& - \left[2\,b + k \right] \left[k_N + b + k^R \right] \left[c_N \left(k_A - b \right) + c_A \left(k_N + b \right) \right] \\
= \; & b \left[b + k \right] \left[c_N \left(k_A + k_N \right) - c_N \left(k_A - b \right) - c_A \left(k_N + b \right) \right] \\
& + \left[2\,b + k \right] \left\{ c_N \left[k_N \left(k_A + k^R \right) + k_A \left(b + k^R \right) \right] \right. \\
& \qquad \left. - \left[k_N + b + k^R \right] \left[c_N \left(k_A - b \right) + c_A \left(k_N + b \right) \right] \right\} \\
= \; & b \left[b + k \right] \left[c_N \, k_N + b \, c_N - c_A \left(k_N + b \right) \right] \\
& + \left[2\,b + k \right] \left\{ c_N \left[k_N \left(k_A + k^R \right) - k_N \left(k_A - b \right) + k_A \left(b + k^R \right) \right. \right. \\
& \qquad \left. \left. - \left(k_A - b \right) \left(b + k^R \right) \right] - c_A \left[k_N + b \right] \left[k_N + b + k^R \right] \right\} \\
= \; & b \left[b + k \right] \left[k_N \left(c_N - c_A \right) + b \left(c_N - c_A \right) \right] \\
& + \left[2\,b + k \right] \left\{ c_N \left[k_N \left(b + k^R \right) + b \left(b + k^R \right) \right] - c_A \left[k_N + b \right] k_N \right. \\
& \qquad \left. - c_A \left[k_N + b \right] \left[b + k^R \right] \right\} \\
= \; & b \left[b + k \right] \left[c_N - c_A \right] \left[k_N + b \right] \\
& + \left[2\,b + k \right] \left\{ \left[c_N - c_A \right] \left[k_N + b \right] \left[b + k^R \right] - c_A \left[k_N + b \right] k_N \right\} \\
= \; & b \left[b + k \right] \left[c_N - c_A \right] \left[k_N + b \right] \\
& + \left[2\,b + k \right] \left[k_N + b \right] \left[\left(b + k^R \right) \left(c_N - c_A \right) - c_A \, k_N \right] . \qquad (152)
\end{aligned}
$$

(140), (150), (151), and (152) imply:

$$
\begin{aligned}
F \; = \; & \left[a \left(b + k \right) + b \, c \right] b \left[b + k \right] \left[b + k_N \right] k_N \\
& + b \left[b + k \right] \left\{ b \left[b + k \right] \left[c_N - c_A \right] \left[k_N + b \right] \right. \\
& \qquad \left. + \left[2\,b + k \right] \left[k_N + b \right] \left[\left(b + k^R \right) \left(c_N - c_A \right) - c_A \, k_N \right] \right\} \\
= \; & b \left[b + k \right] \left[b + k_N \right] \left\{ k_N \left[a \left(b + k \right) + b \, c - \left(2\,b + k \right) c_A \right] \right. \\
& \qquad \left. + \left[c_N - c_A \right] \left[b \, \left(b + k \right) + \left(2\,b + k \right) \left(b + k^R \right) \right] \right\} \\
= \; & \left[b + k_N \right] Y_1 \, . \qquad (153)
\end{aligned}
$$

(141), (149), and (153) imply that $\frac{\partial \bar{p}_2}{\partial k_A} \; = \; \frac{[\, k_N + b \,] \, Y_1}{(D_2)^2} \; > \; 0$.

(140), (142), (148), (149), and (153) imply:

$$
\frac{\partial \left(\bar{p}_3 - \bar{p}_2 \right)}{\partial k_A} \; = \; Y_1 \left[\frac{k_N}{\left(D_3 \right)^2} - \frac{k_N + b}{\left(D_2 \right)^2} \right] \; < \; 0 \, .
$$

The inequality here holds because: (i) $Y_1 > 0$, from (141); (ii) $k_N < k_N + b$; and (iii) $D_3 > D_2 > 0$, from (7) and (8).

To prove that $\frac{\partial(\bar{p}_3 - \bar{p}_2)}{\partial k_N} > 0$ if $k_A - b$ is sufficiently small, we introduce the following:

<u>Definition</u>. $$Y_2 \equiv b[b+k]\{k_A[a(b+k)+bc-(2b+k)c_N] - [c_N - c_A]\left[b(b+k)+(2b+k)\left(b+k^R\right)\right]\}. \tag{154}$$

Observe that:

$$Y_2 > 0. \tag{155}$$

(155) follows from (3) and (154) because:

$$a[b+k]+bc-[2b+k]c_N = a[2b+k]-ab+bc-[2b+k]c_N$$
$$= [a-c_N][2b+k]-b[a-c] \quad \text{and}$$
$$b[b+k]+[2b+k]\left[b+k^R\right] = b^2+bk+2b^2+2bk^R+bk+kk^R$$
$$= 3b^2+2bk+2bk^R+kk^R = 3b^2+2b\left[k+k^R\right]+kk^R.$$

(8) implies:

$$(D_3)^2\frac{\partial\bar{p}_3}{\partial k_N} = D_3\left\{[a(b+k)+bc]\left[b+k^R+k_A\right]+b[b+k]c_A\right\}$$
$$-\left\{b[b+k]+[2b+k]\left[k_A+b+k^R\right]\right\}$$
$$\cdot\{[a(b+k)+bc]\left[\left(b+k^R\right)(k_A+k_N)+k_A k_N\right]$$
$$+\ b\,[b+k][c_N k_A + c_A k_N]\}$$

$$= \left\{[a(b+k)+bc]\left[b+k^R+k_A\right]+b[b+k]c_A\right\}$$
$$\cdot\left\{b[b+k][k_A+k_N]+[2b+k]\left[k_A k_N+(k_A+k_N)\left(b+k^R\right)\right]\right\}$$
$$-\left\{b[b+k]+[2b+k]\left[k_A+b+k^R\right]\right\}$$
$$\cdot\{[a(b+k)+bc]\left[\left(b+k^R\right)(k_A+k_N)+k_A k_N\right]$$
$$+\ b\,[b+k][c_N k_A + c_A k_N]\}$$

$$= [a(b+k)+bc]\left[b+k^R+k_A\right]b[b+k][k_A+k_N]$$
$$+\ [a(b+k)+bc]\left[b+k^R+k_A\right][2b+k]\left[\left(b+k^R\right)(k_A+k_N)+k_A k_N\right]$$
$$+\ b[b+k]c_A b[b+k][k_A+k_N]$$

$$+ \; b\,[\,b+k\,]\,c_A\,[\,2\,b+k\,]\,\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N\,\Big]$$

$$- \; b\,[\,b+k\,]\,[\,a\,(b+k)+b\,c\,]\,\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N\,\Big]$$

$$- \; b\,[\,b+k\,]\,b\,[\,b+k\,]\,[\,c_N\,k_A+c_A\,k_N\,]$$

$$- \; [\,2\,b+k\,]\,\Big[\,k_A+b+k^R\,\Big]\,[\,a\,(b+k)+b\,c\,]\,\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N\,\Big]$$

$$- \; [\,2\,b+k\,]\,\Big[\,k_A+b+k^R\,\Big]\,b\,[\,b+k\,]\,[\,c_N\,k_A+c_A\,k_N\,] \;\equiv\; \Lambda\,. \tag{156}$$

(156) implies:

$$\Lambda \;=\; [\,a\,(b+k)+b\,c\,]\,\Lambda_1+b\,[\,b+k\,]\,\Lambda_2 \tag{157}$$

where

$$\begin{aligned}\Lambda_1 \;\equiv\;& b\,[\,b+k\,]\,\Big[\,b+k^R+k_A\,\Big]\,[\,k_A+k_N\,]\\ &+ [\,2\,b+k\,]\,\Big[\,b+k^R+k_A\,\Big]\,\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N\,\Big]\\ &- b\,[\,b+k\,]\,\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N\,\Big]\\ &- [\,2\,b+k\,]\,\Big[\,b+k^R+k_A\,\Big]\,\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N\,\Big]\\ =\;& b\,[\,b+k\,]\,\Big\{\,\Big[\,b+k^R+k_A\,\Big]\,[\,k_A+k_N\,]-\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N\,\Big]\,\Big\}\\ =\;& b\,[\,b+k\,]\,\{\,k_A\,[\,k_A+k_N\,]-k_A\,k_N\,\} \;=\; b\,[\,b+k\,]\,(k_A)^2 \quad \text{and}\end{aligned} \tag{158}$$

$$\begin{aligned}\Lambda_2 \;\equiv\;& c_A\,b\,[\,b+k\,]\,[\,k_A+k_N\,]+c_A\,[\,2\,b+k\,]\,\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N\,\Big]\\ &-\; b\,[\,b+k\,]\,[\,c_N\,k_A+c_A\,k_N\,]-[\,2\,b+k\,]\,\Big[\,k_A+b+k^R\,\Big]\,[\,c_N\,k_A+c_A\,k_N\,]\\ =\;& -b\,[\,b+k\,]\,k_A\,[\,c_N-c_A\,]+[\,2\,b+k\,]\,\Lambda_3\end{aligned} \tag{159}$$

where

$$\begin{aligned}\Lambda_3 \;\equiv\;& c_A\,\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N\,\Big]-\Big[\,k_A+b+k^R\,\Big]\,[\,c_N\,k_A+c_A\,k_N\,]\\ =\;& c_A\,\Big[\,\big(b+k^R\big)\,(k_A+k_N)+k_A\,k_N-k_N\,\big(k_A+b+k^R\big)\,\Big]-c_N\,k_A\,\Big[\,k_A+b+k^R\,\Big]\\ =\;& c_A\,k_A\,\Big[\,b+k^R\,\Big]-c_N\,k_A\,\Big[\,k_A+b+k^R\,\Big] \;=\; -\,\Big[\,b+k^R\,\Big]\,k_A\,[\,c_N-c_A\,]-c_N\,(k_A)^2\,.\end{aligned} \tag{160}$$

(159) and (160) imply:

$$\begin{aligned}\Lambda_2 \;=\;& -b\,[\,b+k\,]\,k_A\,[\,c_N-c_A\,]-[\,2\,b+k\,]\,\Big\{\,\Big[\,b+k^R\,\Big]\,k_A\,[\,c_N-c_A\,]-c_N\,(k_A)^2\,\Big\}\\ =\;& -k_A\,[\,c_N-c_A\,]\,\Big\{\,b\,[\,b+k\,]+[\,2\,b+k\,]\,\Big[\,b+k^R\,\Big]\,\Big\}-[\,2\,b+k\,]\,c_N\,(k_A)^2\,.\end{aligned} \tag{161}$$

(154), (157), (158), and (161) imply:

$$\begin{aligned}
\Lambda \; = \; & [\, a\,(b+k) + b\,c\,]\, b\,[\, b+k\,]\,(k_A)^2 \\
& - b\,[\, b+k\,]\, k_A\,[\, c_N - c_A\,]\left\{ b\,[\, b+k\,] + [\, 2\,b+k\,]\left[\, b+k^R \right] \right\} \\
& - b\,[\, b+k\,]\,[\, 2\,b+k\,]\, c_N\,(k_A)^2 \\
= \; & b\,[\, b+k\,]\, k_A\,\{\, [\, a\,(b+k) + b\,c\,]\, k_A - [\, 2\,b+k\,]\, c_N\, k_A \\
& \qquad - [\, c_N - c_A\,]\left[\, b\,(b+k) + (2\,b+k)\left(b+k^R\right) \right] \} \\
= \; & b\,[\, b+k\,]\, k_A\,\{\, k_A\,[\, a\,(b+k) + b\,c - (2\,b+k)\, c_N\,] \\
& \qquad - [\, c_N - c_A\,]\left[\, b\,(b+k) + (2\,b+k)\left(b+k^R\right) \right] \} \; = \; k_A\, Y_2\,. \qquad (162)
\end{aligned}$$

(155), (156), and (162) imply that $\frac{\partial \bar{p}_3}{\partial k_N} = \frac{k_A\, Y_2}{(D_3)^2} > 0$.

(7) implies:

$$\begin{aligned}
(D_2)^2\, \frac{\partial \bar{p}_2}{\partial k_N} \; = \; & D_2 \left\{ [\, a\,(b+k) + b\,c\,]\left[\, b+k^R + k_A - b \right] + b\,[\, b+k\,]\, c_A \right\} \\
& - \left\{ b\,[\, b+k\,] + [\, 2\,b+k\,]\left[\, k_A - b + b + k^R \right] \right\} \\
& \cdot \; \{\, [\, a\,(b+k) + b\,c\,]\left[\, \left(b+k^R\right)(k_A + k_N) + k_A\, k_N - b\, k_N \right] \\
& \qquad + \; b\,[\, b+k\,]\,[\, c_N\,(k_A - b) + c_A\,(k_N + b)\,] \} \\
= \; & \left\{ [\, a\,(b+k) + b\,c\,]\left[\, k^R + k_A \right] + b\,[\, b+k\,]\, c_A \right\} \\
& \cdot \{\, b\,[\, b+k\,]\,[\, k_A + k_N\,] + [\, 2\,b+k\,]\left[\, k_N\left(k_A + k^R\right) + k_A\left(b+k^R\right) \right] \} \\
& - \left\{ b\,[\, b+k\,] + [\, 2\,b+k\,]\left[\, k_A + k^R \right] \right\} \\
& \cdot \{\, [\, a\,(b+k) + b\,c\,]\left[\, k_A\,(b + k_N) + k^R\,(k_A + k_N) \right] \\
& \qquad + b\,[\, b+k\,]\,[\, c_N\,(k_A - b) + c_A\,(k_N + b)\,] \} \\
= \; & [\, a\,(b+k) + b\,c\,]\left[\, k^R + k_A \right] b\,[\, b+k\,]\,[\, k_A + k_N\,] \\
& + [\, a\,(b+k) + b\,c\,]\left[\, k^R + k_A \right] [\, 2\,b+k\,]\left[\, k_N\left(k_A + k^R\right) + k_A\left(b+k^R\right) \right] \\
& + b\,[\, b+k\,]\, c_A\, b\,[\, b+k\,]\,[\, k_A + k_N\,] \\
& + b\,[\, b+k\,]\, c_A\,[\, 2\,b+k\,]\left[\, k_N\left(k_A + k^R\right) + k_A\left(b+k^R\right) \right]
\end{aligned}$$

$$- \, b \, [\, b + k \,] \, [\, a \, (b + k) + b \, c \,] \left[\, k_A \, (b + k_N) + k^R \, (k_A + k_N) \,\right]$$

$$- \, b \, [\, b + k \,] \; b \, [\, b + k \,] \, [\, c_N \, (k_A - b) + c_A \, (k_N + b) \,]$$

$$- \, [\, 2 \, b + k \,] \left[\, k_A + k^R \,\right] [\, a \, (b + k) + b \, c \,] \left[\, k_A \, (b + k_N) + k^R \, (k_A + k_N) \,\right]$$

$$- \, [\, 2 \, b + k \,] \left[\, k_A + k^R \,\right] \; b \, [\, b + k \,] \, [\, c_N \, (k_A - b) + c_A \, (k_N + b) \,] \;\equiv\; \Gamma \, . \tag{163}$$

(163) implies:

$$\Gamma \;=\; [\, a \, (b + k) + b \, c \,] \, \Gamma_1 + b \, [\, b + k \,] \, \Gamma_2 \tag{164}$$

where

$$\Gamma_1 \;\equiv\; b \, [\, b + k \,] \left[\, k^R + k_A \,\right] [\, k_A + k_N \,]$$

$$+ \, [\, 2 \, b + k \,] \left[\, k^R + k_A \,\right] \left[\, k_N \left(k_A + k^R\right) + k_A \left(b + k^R\right) \,\right]$$

$$- \, b \, [\, b + k \,] \left[\, k_A \, (b + k_N) + k^R \, (k_A + k_N) \,\right]$$

$$- \, [\, 2 \, b + k \,] \left[\, k_A + k^R \,\right] \left[\, k_A \, (b + k_N) + k^R \, (k_A + k_N) \,\right]$$

$$= \; b \, [\, b + k \,] \left\{ \left[\, k^R + k_A \,\right] [\, k_A + k_N \,] - \left[\, k_A \, (b + k_N) + k^R \, (k_A + k_N) \,\right] \right\}$$

$$= \; b \, [\, b + k \,] \, \{\, k_A \, [\, k_A + k_N \,] - k_A \, [\, b + k_N \,] \,\} \;=\; b \, [\, b + k \,] \, [\, k_A - b \,] \, k_A \tag{165}$$

and

$$\Gamma_2 \;\equiv\; b \, [\, b + k \,] \, [\, k_A + k_N \,] \, c_A + c_A \, [\, 2 \, b + k \,] \left[\, k_N \left(k_A + k^R\right) + k_A \left(b + k^R\right) \,\right]$$

$$- \, b \, [\, b + k \,] \, [\, c_N \, (k_A - b) + c_A \, (k_N + b) \,]$$

$$- \, [\, 2 \, b + k \,] \left[\, k_A + k^R \,\right] [\, c_N \, (k_A - b) + c_A \, (k_N + b) \,]$$

$$= \; b \, [\, b + k \,] \, [\, c_A \, (k_A + k_N) - c_N \, (k_A - b) - c_A \, (k_N + b) \,]$$

$$+ \, [\, 2 \, b + k \,] \; \left\{ \, c_A \left[\, k_N \left(k_A + k^R\right) + k_A \left(b + k^R\right) \,\right] \right.$$

$$\left. - \; \left[\, k_A + k^R \,\right] [\, c_N \, (k_A - b) + c_A \, (k_N + b) \,] \, \right\}$$

$$= \; b \, [\, b + k \,] \, [\, c_A \, (k_A - b) - c_N \, (k_A - b) \,]$$

$$+ \, [\, 2 \, b + k \,] \; \left\{ \, c_A \left[\, k_N \left(k_A + k^R\right) + k_A \left(b + k^R\right) - \left(k_A + k^R\right) (k_N + b) \,\right] \right.$$

$$\left. - \;\; c_N \left[\, k_A + k^R \,\right] [\, k_A - b \,] \, \right\}$$

$$= \; - \, b \, [\, b + k \,] \, [\, k_A - b \,] \, [\, c_N - c_A \,]$$

$$+ \; [\, 2 \, b + k \,] \; \left\{ \, c_A \left[\, k_A \left(b + k^R\right) - b \left(k_A + k^R\right) \,\right] - c_N \left[\, k_A + k^R \,\right] [\, k_A - b \,] \, \right\}$$

$$= \; -b\,[\,b+k\,]\,[\,k_A-b\,]\,[\,c_N-c_A\,]$$

$$+\;[\,2\,b+k\,]\,\big\{\,c_A\,k^R\,[\,k_A-b\,]-c_N\,k^R\,[\,k_A-b\,]-c_N\,k_A\,[\,k_A-b\,]\,\big\}$$

$$= \; [\,k_A-b\,]\,\big\{-b\,[\,b+k\,]\,[\,c_N-c_A\,]-[\,2\,b+k\,]\,k^R\,[\,c_N-c_A\,]-\;[\,2\,b+k\,]\,k_A\,c_N\,\big\}$$

$$= \; -\;[\,k_A-b\,]\,\big\{\,[\,c_N-c_A\,]\,\big[\,b\,(b+k)+(2\,b+k)\,k^R\,\big]+[\,2\,b+k\,]\,k_A\,c_N\,\big\}\,. \qquad (166)$$

(154), (164), (165), and (166) imply:

$$\Gamma \; = \; [\,a\,(b+k)+b\,c\,]\,b\,[\,b+k\,]\,[\,k_A-b\,]\,k_A$$

$$-\;b\,[\,b+k\,]\,[\,k_A-b\,]\;\{\,[\,c_N-c_A\,]\,\big[\,b\,(b+k)+(2\,b+k)\,k^R\,\big]$$

$$+\,[\,2\,b+k\,]\,k_A\,c_N\,\}$$

$$= \; b\,[\,b+k\,]\,[\,k_A-b\,]\;\{\,[\,a\,(b+k)+b\,c\,]\,k_A-[\,c_N-c_A\,]\,\big[\,b\,(b+k)+(2\,b+k)\,k^R\,\big]$$

$$-\;[\,2\,b+k\,]\,k_A\,c_N\,\}$$

$$= \; [\,k_A-b\,]\,b\,[\,b+k\,]\;\{\,k_A\,[\,a\,(b+k)+b\,c-(2\,b+k)\,c_N\,]$$

$$-\;[\,c_N-c_A\,]\,\big[\,b\,(b+k)+(2\,b+k)\,k^R\,\big]\,\}$$

$$= \; [\,k_A-b\,]\;Y_2\,. \qquad (167)$$

(163) and (167) imply that $\frac{\partial \bar{p}_2}{\partial k_N} \; = \; \frac{[\,k_A-b\,]\,Y_2}{(D_2)^2} \; \gtreqless \; 0 \;\Leftrightarrow\; k_A \; \gtreqless \; b$.

(154), (155), (156), (162), (163), and (167) imply:

$$\frac{\partial\,(\bar{p}_3-\bar{p}_2)}{\partial k_N} \; = \; Y_2\left[\,\frac{k_A}{(D_3)^2}-\frac{k_A-b}{(D_2)^2}\,\right] \; > \; 0 \;\; \text{if } k_A-b \text{ is sufficiently small.} \qquad (168)$$

To prove that $\frac{\partial(\bar{p}_3-\bar{p}_2)}{\partial c} > 0$, observe that (7) implies:

$$\frac{\partial \bar{p}_2}{\partial c} \; = \; \frac{b}{D_2}\,\big[\,\big(b+k^R\big)\,(k_A+k_N)+k_N\,(k_A-b)\,\big]$$

$$= \; \frac{b}{D_2}\,\big[\,k_A\,\big(b+k^R\big)+k_N\,\big(b+k^R+k_A-b\big)\,\big]$$

$$= \; \frac{b}{D_2}\,\big[\,k_A\,\big(b+k^R\big)+k_N\,\big(k_A+k^R\big)\,\big] \; > \; 0\,. \qquad (169)$$

Furthermore, (8) implies:

$$\frac{\partial \bar{p}_3}{\partial c} \; = \; \frac{1}{D_3}\;\big\{\,b\,\big[\,\big(b+k^R\big)\,(k_A+k_N)+k_N\,k_A\,\big]\,\big\}$$

$$= \frac{b}{D_3}\left[\left(b+k^R\right)(k_A+k_N)+k_N\,k_A\right]$$

$$= \frac{b}{D_3}\left[k_A\left(b+k^R\right)+k_N\left(k_A+k^R+b\right)\right] \;>\; 0\,. \tag{170}$$

(169) and (170) imply:

$$\frac{\partial\left(\bar{p}_3 \,-\, \bar{p}_2\right)}{\partial c} \;\stackrel{s}{=}\; \frac{k_A\left[b+k^R\right]+k_N\left[k_A+k^R+b\right]}{D_3} - \frac{k_A\left[b+k^R\right]+k_N\left[k_A+k^R\right]}{D_2} \;>\; 0$$

$$\Leftrightarrow \quad \frac{k_A\left[b+k^R\right]+k_N\left[k^R+b+k_A\right]}{D_3} \;>\; \frac{k_A\left[b+k^R\right]+k_N\left[k_A+k^R\right]}{D_2}$$

$$\Leftrightarrow \quad \frac{k_A\left[b+k^R\right]+k_N\left[k^R+b+k_A\right]}{D_2+b\,k_N\,\left[2\,b+k\right]} \;>\; \frac{k_A\left[b+k^R\right]+k_N\left[k_A+k^R\right]}{D_2}$$

$$\Leftrightarrow \quad \frac{Z+b\,k_N}{D_2+b\,k_N\,\left[2\,b+k\right]} \;>\; \frac{Z}{D_2} \quad \text{where } Z \;\equiv\; k_A\left[b+k^R\right]+k_N\left[k_A+k^R\right]. \tag{171}$$

(171) implies:

$$\frac{\partial\left(\bar{p}_3 \,-\, \bar{p}_2\right)}{\partial c} \;>\; 0 \quad\Leftrightarrow\quad Z\,D_2+b\,k_N\,D_2 \;>\; Z\,D_2+Z\,b\,k_N\,\left[2\,b+k\right]$$

$$\Leftrightarrow \quad b\,k_N\,D_2 \;>\; Z\,b\,k_N\,\left[2\,b+k\right] \quad\Leftrightarrow\quad D_2 \;>\; Z\,\left[2\,b+k\right]$$

$$\Leftrightarrow \quad D_2 > \left[k_A\left(b+k^R\right)+k_N\left(k_A+k^R\right)\right]\left[2\,b+k\right]$$

$$\Leftrightarrow \quad b\left[b+k\right]\left[k_N+k_A\right]+k_N\left[k_A-b\right]\left[2\,b+k\right]+\left[k_N+k_A\right]\left[2\,b+k\right]\left[b+k^R\right]$$

$$> \;\left[k_A\left(b+k^R\right)+k_N\left(k_A+k^R\right)\right]\left[2\,b+k\right]$$

$$\Leftrightarrow \quad b\left[b+k\right]\left[k_N+k_A\right]+k_N\left[k_A-b\right]\left[2\,b+k\right]+\left[k_N+k_A\right]\left[2\,b+k\right]\left[b+k^R\right]$$

$$> \;\left[\left(b+k^R\right)(k_A+k_N)+k_N\,(k_A-b)\right]\left[2\,b+k\right]$$

$$\Leftrightarrow \quad b\left[b+k\right]\left[k_N+k_A\right] \;>\; 0\,. \quad \blacksquare$$

Recall that welfare is:

$$W(\bar{p}) \;\equiv\; S(\bar{p})-d\left[\,\bar{p}\;q_A+(\,a-b\left[\,q_A+q_N+q\,\right])\,q_N\,\right] \;=\; S(\bar{p})-d\,V(\bar{p}) \tag{172}$$

where $d > 0$ is a parameter and $S(\cdot)$ denotes consumer surplus. The gross value that consumers derive from Q units of output is:

$$\frac{1}{2}\left[\,a-P(Q)\,\right]Q+P(Q)\,Q \;=\; \frac{1}{2}\left[\,a+P(Q)\,\right]Q \;=\; \frac{1}{2}\left[\,a+a-b\,Q\,\right]Q \;=\; a\,Q-\frac{b}{2}\,Q^2.$$

Therefore, consumer surplus when the price cap is $\bar{p}$ is:

$$S(\bar{p}) \;=\; a\,Q - \frac{b}{2}\,Q^2 - \bar{p}\,q_A - P(Q)\,[\,q_N + q\,]\,. \tag{173}$$

Lemma 1. *For* $\bar{p} \in (\bar{p}_2, \bar{p}_3)$, $S(\bar{p})$ *is a strictly decreasing, strictly convex function of* $\bar{p}$.

Proof. (57) implies that when $\bar{p} \in (\bar{p}_2, \bar{p}_3)$, so $P(Q) = \bar{p}$:

$$Q \;=\; \frac{a - \bar{p}}{b} \;\;\Rightarrow\;\; \frac{\partial Q}{\partial \bar{p}} \;=\; -\,\frac{1}{b}\,. \tag{174}$$

(173) and (174) imply that for $\bar{p} \in (\bar{p}_2, \bar{p}_3)$, where $P(Q) = \bar{p}$:

$$\frac{\partial S(\bar{p})}{\partial \bar{p}} \;=\; a\,\frac{\partial Q}{\partial \bar{p}} - b\,Q\,\frac{\partial Q}{\partial \bar{p}} - Q - \bar{p}\,\frac{\partial Q}{\partial \bar{p}} \;=\; -\,\frac{a}{b} + Q - Q + \frac{\bar{p}}{b}$$

$$=\; -\,\frac{a - \bar{p}}{b} \;<\; 0 \;\;\Rightarrow\;\; \frac{\partial^2 S(\bar{p})}{\partial\,(\bar{p})^2} \;=\; \frac{1}{b} \;>\; 0\,. \;\blacksquare \tag{175}$$

Lemma 2. $V(\bar{p}_1) \;<\; V(\bar{p}_3)$.

Proof. Lemmas A1 and A3 imply that because $q_A(\bar{p}_1) = 0$ and $P(Q(\bar{p}_3)) = \bar{p}_3$:

$$V(\bar{p}_1) \;=\; \bar{p}_1\,q_N(\bar{p}_1) \;=\; \bar{p}_1\,\frac{[\,a - c_N\,]\,[\,2\,b + k\,] - b\,[\,a - c\,]}{[\,2\,b + k_N + k^R\,]\,[\,2\,b + k\,] - b^2}\,;$$

$$V(\bar{p}_3) \;=\; \bar{p}_3\,Q^R(\bar{p}_3) \;=\; \bar{p}_3\,\frac{[\,b + k\,]\,[\,a - \bar{p}_3\,] - b\,[\,\bar{p}_3 - c\,]}{b\,[\,b + k\,]}\,. \tag{176}$$

Definition. $D_N \;\equiv\; \left[\,2\,b + k_N + k^R\,\right]\,[\,2\,b + k\,] - b^2\,.$ (177)

Because $\bar{p}_1 \,<\, \bar{p}_3$, (176) and (177) imply that $V(\bar{p}_1) \,<\, V(\bar{p}_3)$ if:

$$q_N(\bar{p}_1) \;=\; \frac{[\,a - c_N\,]\,[\,2\,b + k\,] - b\,[\,a - c\,]}{D_N} \;<\; \frac{[\,b + k\,]\,[\,a - \bar{p}_3\,] - b\,[\,\bar{p}_3 - c\,]}{b\,[\,b + k\,]} \;=\; Q^R(\bar{p}_3)$$

$$\Leftrightarrow\;\; \frac{a\,[\,b + k\,] + a\,b - c_N\,[\,2\,b + k\,] - b\,a + b\,c}{D_N} \;<\; \frac{[\,b + k\,]\,a - [\,b + k\,]\,\bar{p}_3 - b\,\bar{p}_3 + b\,c}{b\,[\,b + k\,]}$$

$$\Leftrightarrow\;\; \frac{a\,[\,b + k\,] + b\,c - c_N\,[\,2\,b + k\,]}{D_N} \;<\; \frac{[\,b + k\,]\,a + b\,c - [\,2\,b + k\,]\,\bar{p}_3}{b\,[\,b + k\,]}$$

$$\Leftrightarrow\;\; \frac{a\,[\,b + k\,] + b\,c - c_N\,[\,2\,b + k\,]}{D_N}\,b\,[\,b + k\,] - [\,b + k\,]\,a - b\,c \;<\; -\,[\,2\,b + k\,]\,\bar{p}_3$$

$$\Leftrightarrow \frac{a[b+k]+bc}{2b+k} - \frac{a[b+k]+bc-c_N[2b+k]}{[2b+k]D_N} \, b[b+k] \; > \; \bar{p}_3$$

$$\Leftrightarrow \frac{a[b+k]+bc}{2b+k} - \frac{[a(b+k)+bc]\,b\,[b+k] - c_N[2b+k]\,b\,[b+k]}{[2b+k]D_N} \; > \; \bar{p}_3$$

$$\Leftrightarrow \frac{1}{[2b+k]D_N}\left\{[a(b+k)+bc]\left[\left(2b+k_N+k^R\right)(2b+k) - b^2 - b\,(b+k)\right] + c_N[2b+k]\,b\,[b+k]\right\} \; > \; \bar{p}_3$$

$$\Leftrightarrow \frac{1}{[2b+k]D_N}\left\{[a(b+k)+bc]\left[\left(2b+k_N+k^R\right)(2b+k) - b\,(2b+k)\right] + c_N[2b+k]\,b\,[b+k]\right\} \; > \; \bar{p}_3$$

$$\Leftrightarrow \frac{[a(b+k)+bc]\left[2b+k_N+k^R-b\right] + c_N\,b\,[b+k]}{D_N} \; > \; \bar{p}_3$$

$$\Leftrightarrow \frac{[a(b+k)+bc]\left[b+k_N+k^R\right] + c_N\,b\,[b+k]}{D_N} \; > \; \bar{p}_3\,. \tag{178}$$

(8) implies:

$$\bar{p}_3 = \frac{[a(b+k)+bc]\left[\left(b+k^R\right)(k_N+k_A)+k_N\,k_A\right] + b\,c_N\,[b+k]\,k_A + b\,k_N\,[b+k]\,c_A}{b\,[b+k]\,[k_N+k_A] + k_N\,k_A\,[2b+k] + [k_N+k_A]\,[2b+k]\,[b+k^R]}\,. \tag{179}$$

As established in the proof of Proposition 4 (just below (148)), $\bar{p}_3$ is increasing in k_A. Therefore, (179) implies that because $k_A \leq k_N$ by assumption:

$$\bar{p}_3 \leq \frac{[a(b+k)+bc]\left[2\,k_N\left(b+k^R\right)+(k_N)^2\right] + b\,c_N\,[b+k]\,k_N + b\,k_N\,[b+k]\,c_A}{2\,b\,[b+k]\,k_N + (k_N)^2\,[2b+k] + 2\,k_N\,[2b+k]\,[b+k^R]}\,. \tag{180}$$

(8) implies that $\bar{p}_3$ is increasing in c_A. Therefore, because $c_A \leq c_N$ by assumption, (180) implies:

$$\begin{aligned}
\bar{p}_3 \;&\leq\; \frac{[a(b+k)+bc]\left[2\,k_N\left(b+k^R\right)+(k_N)^2\right] + 2\,b\,c_N\,[b+k]\,k_N}{2\,b\,[b+k]\,k_N + (k_N)^2\,[2b+k] + 2\,k_N\,[2b+k]\,[b+k^R]} \\
&=\; \frac{[a(b+k)+bc]\left[2\left(b+k^R\right)+k_N\right] + 2\,b\,c_N\,[b+k]}{2\,b\,[b+k] + k_N\,[2b+k] + 2\,[2b+k]\,[b+k^R]} \\
&=\; \frac{[a(b+k)+bc]\left[b+k^R+\frac{k_N}{2}\right] + b\,c_N\,[b+k]}{b\,[b+k] + \frac{k_N}{2}\,[2b+k] + [2b+k]\,[b+k^R]} \\
&=\; \frac{[a(b+k)+bc]\left[b+k^R+\frac{k_N}{2}\right] + b\,c_N\,[b+k]}{[2b+k]\left[b+k^R+\frac{k_N}{2}\right] + b\,[b+k]}
\end{aligned}$$

$$= \frac{[\, a\,(b+k) + b\,c\,] \left[\, b + k^R + \frac{k_N}{2} \,\right] + b\,c_N\,[\, b+k\,]}{[\, 2\,b + k\,] \left[\, 2\,b + k^R + \frac{k_N}{2} \,\right] - b^2} \,. \tag{181}$$

The last equality in (181) holds because:

$$\begin{aligned} & [\, 2\,b + k\,] \left[b + k^R + \frac{k_N}{2} \right] + b\,[\, b + k\,] \\ & \quad = [\, 2\,b + k\,] \left[2\,b + k^R + \frac{k_N}{2} \right] - b\,[\, 2\,b + k\,] + b\,[\, b+k\,] \\ & \quad = [\, 2\,b + k\,] \left[2\,b + k^R + \frac{k_N}{2} \right] - 2\,b^2 - b\,k + b^2 + b\,k \\ & \quad = [\, 2\,b + k\,] \left[2\,b + k^R + \frac{k_N}{2} \right] - b^2. \end{aligned}$$

(177), (178), and (181) imply that the Lemma holds if:

$$\begin{aligned} & \frac{[\, a\,(b+k) + b\,c\,] \left[\, b + k^R + \frac{k_N}{2} \,\right] + b\,c_N\,[\, b+k\,]}{[\, 2\,b + k\,] \left[\, 2\,b + k^R + \frac{k_N}{2} \,\right] - b^2} \\ & \qquad < \frac{[\, a\,(b+k) + b\,c\,] \left[\, b + k^R + k_N \,\right] + b\,c_N\,[\, b+k\,]}{[\, 2\,b + k\,]\,[\, 2\,b + k^R + k_N\,] - b^2} \,. \end{aligned} \tag{182}$$

<u>Definition</u>. $f(x) \equiv \dfrac{A \left[\, b + k^R + x \,\right] + b\,c_N\,[\, b+k\,]}{[\, 2\,b + k\,]\,[\, 2\,b + k^R + x\,] - b^2}$ where $A \equiv a\,[\, b+k\,] + b\,c$. (183)

(183) implies that (182) holds if $\frac{\partial f}{\partial x} > 0$. (177) and (183) imply:

$$\begin{aligned} \frac{\partial f(\cdot)}{\partial x} & \overset{s}{=} \left\{ [\, 2\,b + k\,] \left[\, 2\,b + k^R + x \,\right] - b^2 \right\} A \\ & \qquad - [\, 2\,b + k\,] \left\{ A \left[\, b + k^R + x \,\right] + b\,c_N\,[\, b + k\,] \right\} \\ & = A \left\{ [\, 2\,b + k\,] \left[\, 2\,b + k^R + x - \left(b + k^R + x \right) \right] - b^2 \right\} - b\,[\, b + k\,]\,[\, 2\,b + k\,]\,c_N \\ & = A \left\{ b\,[\, 2\,b + k\,] - b^2 \right\} - b\,[\, b+k\,]\,[\, 2\,b + k\,]\,c_N \\ & = A\,b\,[\, b + k\,] - b\,[\, b+k\,]\,[\, 2\,b + k\,]\,c_N \overset{s}{=} A - [\, 2\,b + k\,]\,c_N \\ & = a\,[\, b + k\,] + b\,c - [\, 2\,b + k\,]\,c_N > 0\,. \end{aligned}$$

The inequality here holds because (3) implies:

$$[\, a - c_N\,]\,[\, 2\,b + k\,] - b\,[\, a - c\,] > 0$$

$$\Rightarrow \; [\,a - c_N\,][\,b + k\,] + b\,[\,a - c_N\,] - b\,[\,a - c\,] \; > \; 0$$

$$\Rightarrow \; [\,a - c_N\,][\,b + k\,] + b\,[\,c - c_N\,] \; > \; 0$$

$$\Rightarrow \; a\,[\,b + k\,] - c_N\,[\,b + k\,] + b\,[\,c - c_N\,] \; > \; 0$$

$$\Rightarrow \; a\,[\,b + k\,] + b\,c - [\,2\,b + k\,]\,c_N \; > \; 0\,. \;\blacksquare$$

Proposition 5. $\bar{p}^* \in [\,\bar{p}_1, \bar{p}_2\,]$.

Proof. Proposition 3 and Lemma 1 imply that $W(\cdot)$ is a strictly convex function of $\bar{p}$ for $\bar{p} \in (\bar{p}_2, \bar{p}_3)$. Therefore, $\bar{p}^* \notin (\bar{p}_2, \bar{p}_3)$. Lemma A1 implies that $W(\bar{p}) = W(\bar{p}_1)$ for all $\bar{p} < \bar{p}_1$. Lemma A4 implies that $W(\bar{p}) = W(\bar{p}_3)$ for all $\bar{p} > \bar{p}_3$. Therefore, $\bar{p}^* \in [\,\bar{p}_1, \bar{p}_2\,] \bigcup \bar{p}_3$.

It remains to show that $\bar{p}^* \neq \bar{p}_3$. The proof of Lemma 2 establishes that:

$$Q^R(\bar{p}_1) \; < \; Q^R(\bar{p}_3) \tag{184}$$

where $Q^R(\bar{p})$ is R's total output when the price cap is $\bar{p}$. Lemma A6 and Proposition 2 imply:

$$Q^R(\bar{p}_3) \; < \; Q^R(\bar{p}_2)\,. \tag{185}$$

(184) and (185) imply that $Q^R(\bar{p}_1) < Q^R(\bar{p}_3) < Q^R(\bar{p}_2)$. $Q^R(\bar{p})$ is continuous and monotonically increasing in $\bar{p}$ for $\bar{p} \in (\bar{p}_1, \bar{p}_2)$ (from Lemma A2). Therefore, the intermediate value theorem implies that there exists a $\bar{p}_E \in (\bar{p}_1, \bar{p}_2)$ such that:

$$Q^R(\bar{p}_E) \; = \; Q^R(\bar{p}_3)\,. \tag{186}$$

(12) implies that the rival's output q is determined by:

$$a - b\left[\,Q^R(\bar{p}) + q(\bar{p})\,\right] - c - b\,q(\bar{p}) - k\,q(\bar{p}) \; = \; 0\,. \tag{187}$$

(186) and (187) imply:

$$q(\bar{p}_E) \; = \; q(\bar{p}_3)\,. \tag{188}$$

(186) and (188) imply:

$$Q(\bar{p}_E) \; = \; Q(\bar{p}_3) \;\text{ and }\; P(Q(\bar{p}_E)) \; = \; P(Q(\bar{p}_3))\,. \tag{189}$$

R's revenue is:

$$\begin{aligned} V_2(\bar{p}_E) \; &= \; \bar{p}_E\, q_A(\bar{p}_E) + P(Q(\bar{p}_E))\, q_N(\bar{p}_E) \\ &< \; P(Q(\bar{p}_E))\, q_A(\bar{p}_E) + P(Q(\bar{p}_E))\, q_N(\bar{p}_E) \\ &= \; P(Q(\bar{p}_E))\, Q^R(\bar{p}_E) \; = \; P(Q(\bar{p}_3))\, Q^R(\bar{p}_3) \; = \; V_3(\bar{p}_3)\,. \end{aligned} \tag{190}$$

The inequality in (190) holds because $\bar{p}_E < P(Q(\bar{p}_E))$, since $\bar{p}_E \in (\bar{p}_1, \bar{p}_2)$. The penultimate equality in (190) reflects (189). The last equality in (190) holds because $P(Q(\bar{p}_3)) = \bar{p}_3$.

(173) and (189) imply:

$$
\begin{aligned}
S(\bar{p}_E) &= a\,Q(\bar{p}_E) - \frac{b}{2}\,Q(\bar{p}_E)^2 - P(Q(\bar{p}_E))\,[\,q(\bar{p}_E) + q_N(\bar{p}_E)\,] - \bar{p}_E\,q_A(\bar{p}_E) \\
&> a\,Q(\bar{p}_E) - \frac{b}{2}\,Q(\bar{p}_E)^2 - P(Q(\bar{p}_E))\,[\,q(\bar{p}_E) + q_N(\bar{p}_E) + q_A(\bar{p}_E)\,] \\
&= a\,Q(\bar{p}_E) - \frac{b}{2}\,Q(\bar{p}_3)^2 - P(Q(\bar{p}_E))\,Q(\bar{p}_E) \\
&= a\,Q(\bar{p}_E) - \frac{b}{2}\,Q(\bar{p}_3)^2 - P\,(Q(\bar{p}_3))\,Q(\bar{p}_3) \;=\; S(\bar{p}_3)\,. \qquad (191)
\end{aligned}
$$

The inequality in (191) holds because $\bar{p}_E < P\,(Q(\bar{p}_E))$, since $\bar{p}_E \in (\bar{p}_1, \bar{p}_2)$. (190) and (191) imply that consumer surplus is higher and R's revenue is lower when $\bar{p} = \bar{p}_E$ than when $\bar{p} = \bar{p}_3$. Therefore, $W(\bar{p}_E) > W(\bar{p}_3)$, so $\bar{p}^* \neq \bar{p}_3$. ■

Lemma 3. *In equilibrium, for* $\bar{p} \in (\bar{p}_1, \bar{p}_2)$, $\frac{dq_A}{d\bar{p}} > 0$, $\frac{dq_N}{d\bar{p}} < 0$, $\frac{dq}{d\bar{p}} < 0$, $\frac{dQ^R}{d\bar{p}} > 0$, $\frac{dQ}{d\bar{p}} > 0$, *and* $\frac{dP(Q)}{d\bar{p}} < 0$.

Proof. (2) and (20) – (23) imply that for $\bar{p} \in (\bar{p}_1, \bar{p}_2)$:

$$
\begin{aligned}
\frac{dq_A}{d\bar{p}} &= \frac{3\,b^2 + 2\,b\left[\,k + k_N + k^R\,\right] + k\left[\,k_N + k^R\,\right]}{D} \;>\; 0\,; \\
\frac{dq_N}{d\bar{p}} &= -\,\frac{b\left[\,b + 2\,k^R\,\right] + k\left[\,b + k^R\,\right]}{D} \;<\; 0\,; \quad \frac{dQ^R}{d\bar{p}} \;=\; \frac{[\,2\,b + k\,]\,[\,b + k_N\,]}{D} \;>\; 0\,; \\
\frac{dq}{d\bar{p}} &= -\,\frac{b\,[\,b + k_N\,]}{D} \;<\; 0\,; \quad \text{and} \quad \frac{dQ}{d\bar{p}} \;=\; \frac{[\,b + k\,]\,[\,b + k_N\,]}{D} \;>\; 0\,. \;■ \qquad (192)
\end{aligned}
$$

Lemma 4. *For* $\bar{p} \in (\bar{p}_1, \bar{p}_2)$: *(i)* $V(\bar{p})$ *is a strictly convex function of* $\bar{p}$; *(ii)* $\frac{\partial V(\bar{p})}{\partial \bar{p}} \lesseqgtr 0 \Leftrightarrow \bar{p} \lesseqgtr \bar{p}_{V_2 m}$ *where* $\bar{p}_{V_2 m} \in [\,\bar{p}_1, \bar{p}_2)$; *and (iii)* $\bar{p}_{V_2 m} > \bar{p}_1$ *if* $\Phi_2 \geq 0$, *where*

$$
\begin{aligned}
\Phi_2 \;\equiv\; &\{\,k^R\,[\,2\,b + k\,]\left[\,k^R\,(2\,b + k) + 2\,b\,(3\,b + 2\,k)\,\right] \\
&\quad +\; k_N\,[\,2\,b + k\,]\left[\,k^R\,(2\,b + k) + b^2\,\right] + b^2\left[\,5\,b^2 + 6\,b\,k + 2\,k^2\,\right]\}\;c_N \\
&- \left\{\,b\,[\,3\,b + 2\,k\,] + [\,2\,b + k\,]\left[\,k_N + k^R\,\right]\right\}^2 c_A \\
&- b\left[\,b^2 - k\,k_N + (2\,b + k)\,k^R\,\right][\,a\,(b + k) + b\,c\,]\,. \qquad (193)
\end{aligned}
$$

Corollary to Lemma 4. $\left.\frac{\partial V(\bar{p})}{\partial \bar{p}}\right|_{\bar{p} = \bar{p}_1} < 0$ *if* $\Phi_2 \geq 0$.

Proof of Lemma 4 and its Corollary.

Define:

$$\widetilde{V}_2(\bar{p}) \;\equiv\; q_{A2}(\bar{p})\,\bar{p} + q_{N2}(\bar{p})\,P(Q_2(\bar{p})) \tag{194}$$

where $q_{A2}(\bar{p})$ and $q_{N2}(\bar{p})$ are as defined in (20) and (21), respectively. Observe that $\widetilde{V}_2(\bar{p}) = V(\bar{p})$ for $\bar{p} \in [\,\bar{p}_1, \bar{p}_2\,]$.

Because $P(Q_2) = a - b\,Q_2$, (194) implies:

$$\frac{\partial \widetilde{V}_2(\bar{p})}{\partial \bar{p}} \;=\; q_{A2} + \bar{p}\,\frac{\partial q_{A2}}{\partial \bar{p}} + P(Q_2)\,\frac{\partial q_{N2}}{\partial \bar{p}} - b\,q_{N2}\,\frac{\partial Q_2}{\partial \bar{p}}\,. \tag{195}$$

(2) and Lemma A2 imply:

$$\frac{\partial^2 q_{A2}}{\partial\,(\bar{p})^2} \;=\; \frac{\partial^2 q_{N2}}{\partial\,(\bar{p})^2} \;=\; \frac{\partial^2 q_2}{\partial\,(\bar{p})^2} \;=\; \frac{\partial^2 Q_2}{\partial\,(\bar{p})^2} \;=\; 0\,. \tag{196}$$

(195) and (196) imply:

$$\begin{aligned}\frac{\partial^2 \widetilde{V}_2(\bar{p})}{\partial\,(\bar{p})^2} \;&=\; \frac{\partial q_{A2}}{\partial \bar{p}} + \frac{\partial q_{A2}}{\partial \bar{p}} - b\,\frac{\partial Q_2}{\partial \bar{p}}\,\frac{\partial q_{N2}}{\partial \bar{p}} - b\,\frac{\partial q_{N2}}{\partial \bar{p}}\,\frac{\partial Q_2}{\partial \bar{p}} \\ &=\; 2\,\frac{\partial q_{A2}}{\partial \bar{p}} - 2\,b\,\frac{\partial Q_2}{\partial \bar{p}}\,\frac{\partial q_{N2}}{\partial \bar{p}} \;>\; 0\,. \end{aligned} \tag{197}$$

The inequality in (197) holds because $D > 0$ by assumption, so $\frac{\partial q_{A2}}{\partial \bar{p}} > 0$ from (20), $\frac{\partial Q_2}{\partial \bar{p}} > 0$ from (24), and $\frac{\partial q_{N2}}{\partial \bar{p}} < 0$ from (21).

$\bar{p}_{V_2 m} \;\equiv\; \arg\min_{\bar{p}} \{\, \widetilde{V}_2(\bar{p})\,\}$ is unique and is determined by:

$$\frac{\partial \widetilde{V}_2(\bar{p}_{V_2 m})}{\partial \bar{p}} \;\equiv\; \left.\frac{\partial \widetilde{V}_2(\bar{p})}{\partial \bar{p}}\right|_{\bar{p}\,=\,\bar{p}_{V_2 m}} \;=\; 0. \tag{198}$$

This is the case because (2), (20) – (24), and (195) imply that $\frac{\partial \widetilde{V}_2(\bar{p})}{\partial \bar{p}}$ is a linear function of $\bar{p}$. Therefore, $\widetilde{V}_2(\bar{p})$ is a quadratic function of $\bar{p}$. Consequently, (197) implies that $\widetilde{V}_2(\bar{p})$ has a unique minimum that is determined by (198).

To prove the Corollary to Lemma 4 and thereby establish that $\bar{p}_{V_2 m} > \bar{p}_1$ when $\Phi_2 \geq 0$, observe that R's revenue is:

$$V(\bar{p}) \;=\; \bar{p}\,q_A + P(Q)\,q_N \;=\; \bar{p}\,q_A + [\,a - b\,Q\,]\,q_N\,. \tag{199}$$

(199) implies that the Corollary to Lemma 4 holds if:

$$\frac{\partial^+ V(\bar{p}_1)}{\partial \bar{p}} \;=\; q_A + \bar{p}_1\,\frac{\partial q_A}{\partial \bar{p}} - b\,\frac{\partial Q}{\partial \bar{p}}\,q_N + P(Q)\,\frac{\partial q_N}{\partial \bar{p}} \;<\; 0\,, \tag{200}$$

where: (i) $\frac{\partial^+ V(\bar{p}_1)}{\partial \bar{p}} = \left.\frac{\partial^+ V(\bar{p})}{\partial \bar{p}}\right|_{\bar{p}\,=\,\bar{p}_1}$ denotes the right-sided derivative of $V(\cdot)$; (ii) $\frac{\partial q_A}{\partial \bar{p}}$, $\frac{\partial q_N}{\partial \bar{p}}$, and $\frac{\partial Q}{\partial \bar{p}}$ pertain to the quantities identified in Lemma A2 (which prevail when $\bar{p} \in (\bar{p}_1, \bar{p}_2)$);

and (iii) q_A, q_N, and Q are as defined in Lemma A1.

Define:

$$
\begin{aligned}
E &= 2\,b\,[\,2\,b+k\,] + \left[\,k_N + k^R\,\right][\,2\,b+k\,] - b^2 \\
&= 3\,b^2 + 2\,b\,k + \left[\,k_N + k^R\,\right][\,2\,b+k\,] \\
&= b\,[\,3\,b+2\,k\,] + [\,2\,b+k\,]\left[\,k_N + k^R\,\right]. \qquad (201)
\end{aligned}
$$

(201) and Lemma A2 imply that when $\bar{p} \in (\bar{p}_1, \bar{p}_2)$:

$$
\begin{aligned}
\frac{\partial q_N}{\partial \bar{p}} &= -\,\frac{b\,k + 2\,b\,k^R + k\,k^R + b^2}{D}\,; \\
\frac{\partial q_A}{\partial \bar{p}} &= \frac{2\,b\,k + 2\,b\,k_N + 2\,b\,k^R + k\,k_N + k\,k^R + 3\,b^2}{D} \\
&= \frac{\left[\,2\,b + k_N + k^R\,\right]\,[\,2\,b+k\,] - b^2}{D} \;=\; \frac{E}{D}\,; \\
\frac{\partial Q}{\partial \bar{p}} &= \frac{1}{D}\left\{\,2\,b\,k + 2\,b\,k_N + 2\,b\,k^R + k\,k_N + k\,k^R + 3\,b^2 \right. \\
&\qquad \left. -\left[\,b\,k + 2\,b\,k^R + k\,k^R + b^2\,\right] - \left[\,b^2 + k_N\,b\,\right]\right\} \\
&= \frac{b\,k + b\,k_N + k\,k_N + b^2}{D} \;=\; \frac{[\,b+k\,]\,[\,b+k_N\,]}{D}\,. \qquad (202)
\end{aligned}
$$

Lemma A1 implies that when $\bar{p} \leq \bar{p}_1$:

$$
q_N \;=\; \frac{[\,a-c_N\,]\,[\,2\,b+k\,] - b\,[\,a-c\,]}{E}\,, \quad q \;=\; \frac{[\,a-c\,]\left[\,2\,b+k_N+k^R\,\right] - b\,[\,a-c_N\,]}{E}, \text{ and}
$$

$$
\begin{aligned}
P(Q) &= a - b\,[\,q_N + q\,] \;=\; a - b\,\frac{[\,a-c_N\,]\,[\,b+k\,] + \left[\,b+k_N+k^R\,\right][\,a-c\,]}{E} \\
&= \frac{a\,E - b\,[\,a-c_N\,]\,[\,b+k\,] - b\,\left[\,b+k_N+k^R\,\right][\,a-c\,]}{E}\,. \qquad (203)
\end{aligned}
$$

(200) – (203) imply that because $q_A = 0$ when $\bar{p} = \bar{p}_1$ (from Lemma A1):

$$
\begin{aligned}
\frac{\partial^+ V(\bar{p}_1)}{\partial \bar{p}} &= \bar{p}_1\,\frac{E}{D} - b\left[\frac{(b+k)\,(b+k_N)}{D}\right]\left[\frac{(a-c_N)\,(2\,b+k) - b\,(a-c)}{E}\right] \qquad (204) \\
&\quad - \left[\frac{a\,E - b\,(a-c_N)\,(b+k) - b\,\left(b+k_N+k^R\right)(a-c)}{E}\right] \\
&\quad \cdot \left[\frac{b\,k + 2\,b\,k^R + k\,k^R + b^2}{D}\right]
\end{aligned}
$$

$$= \frac{1}{D\,E} \{ \, \bar{p}_1 \, E^2 - b \, [\, b + k \,] \, [\, b + k_N \,] \, [\, (a - c_N) \, (2\, b + k) - b \, (a - c\,) \,] - \left[\, a\, E - b \, (a - c_N) \, (b + k) - b \, \left(b + k_N + k^R \right) (a - c\,) \right] \cdot \left[\, b\, k + 2\, b\, k^R + k\, k^R + b^2 \,\right] \} \, . \tag{205}$$

(6) and (201) imply:

$$E \; \bar{p}_1 \; = \; c_A \, E + [\, a - c_N \,] \, [\, 2\, b + k \,] \left[\, b + k^R \,\right] - b \, [\, a - c \,] \left[\, b + k^R \,\right] . \tag{206}$$

(201), (205), and (206) imply:

$$\frac{\partial^+ V(\bar{p}_1)}{\partial \bar{p}} = \frac{1}{D\,E} \{ \, c_A \, E^2 + E \left[\, (a - c_N) \, (2\, b + k) \left(b + k^R \right) - b \, (a - c) \left(b + k^R \right) \right] - b \, [\, b + k \,] \, [\, b + k_N \,] \, [\, (a - c_N) \, (2\, b + k) - b \, (a - c) \,] - \left[\, a\, E - b \, (a - c_N) \, (b + k) - b \, \left(b + k_N + k^R \right) (a - c) \right] \left[\, b\, k + 2\, b\, k^R + k\, k^R + b^2 \,\right] \}$$

$$= \frac{1}{D\,E} \{ \, c_A \, E^2 + \left[\; E \, \left(b + k^R \right) - b \, (b + k) \, (b + k_N) \right] [\, (a - c_N) \, (2\, b + k) - b \, (a - c) \,] - \left[\, a\, E - b \, (a - c_N) \, (b + k) - b \, \left(b + k_N + k^R \right) (a - c) \right] \left[\, b\, k + 2\, b\, k^R + k\, k^R + b^2 \,\right] \}$$

$$= \frac{1}{D\,E} \left[\, c_A \, E^2 - \tilde{E} \,\right] . \tag{207}$$

where

$$\tilde{E} = \left[\, a\, E - b \, (a - c_N) \, (b + k) - b \, \left(b + k_N + k^R \right) (a - c) \right] \left[\, b\, k + 2\, b\, k^R + k\, k^R + b^2 \,\right] - \left[\, E \left(b + k^R \right) - b \, (b + k) \, (b + k_N) \right] [\, (a - c_N) \, (2\, b + k) - b \, (a - c) \,]$$

$$= \left[\, a\, E - b \, (a - c_N) \, (b + k) - b \, \left(b + k_N + k^R \right) (a - c) \right] \left[\, b\, k + 2\, b\, k^R + k\, k^R + b^2 \,\right] + b \, [\, b + k \,] \, [\, b + k_N \,] \, [\, (a - c_N) \, (2\, b + k) - b \, (a - c) \,] - E \, \left[\, b + k^R \,\right] [\, a - c_N \,] \, [\, 2\, b + k \,] + E \, \left[\, b + k^R \,\right] \, b \, [\, a - c \,]$$

$$= \left[\, a\, E - b \, (a - c_N) \, (b + k) - b \, \left(b + k_N + k^R \right) (a - c) \right] \left[\, b\, k + 2\, b\, k^R + k\, k^R + b^2 \,\right] + b \, [\, b + k \,] \, [\, b + k_N \,] \, [\, (a - c_N) \, (2\, b + k) - b \, (a - c) \,] - E \left[\, b + k^R \,\right] \, a \, [\, 2\, b + k \,] + E \, \left[\, b + k^R \,\right] b \, [\, a - c \,] + c_N \, E \, \left[\, b + k^R \,\right] \, [\, 2\, b + k \,] . \tag{208}$$

(201) implies:

$$\left[\, b + k^R \,\right] [\, 2\, b + k \,] = \left[\, 2\, b + k_N + k^R \,\right] [\, 2\, b + k \,] - [\, b + k_N \,] \, [\, 2\, b + k \,] = E + b^2 - [\, b + k_N \,] \, [\, 2\, b + k \,] = E - \left[\, (b + k_N) \, (2\, b + k) - b^2 \,\right] . \tag{209}$$

(201), (208), and (209) imply:

$$\tilde{E} \; = \; \hat{E} + c_N \, E^2, \;\; \text{where}$$

$$\hat{E} \; \equiv \; \left[\, a \, E - b \, (a - c_N) \, (b + k) - b \left(b + k_N + k^R \right) (a - c) \, \right] \left[\, b \, k + 2 \, b \, k^R + k \, k^R + b^2 \, \right]$$

$$+ \; b \, [\, b + k \,] \, [\, b + k_N \,] \, [\, (a - c_N) \, (2 \, b + k) - b \, (a - c) \,]$$

$$- \; E \left[\, b + k^R \right] \, a \, [\, 2 \, b + k \,] + E \, \left[\, b + k^R \right] b \, [\, a - c \,]$$

$$- \; E \left[\, (b + k_N) \, (2 \, b + k) - b^2 \, \right] c_N \, . \tag{210}$$

(207) and (210) imply:

$$\frac{\partial^+ V(\bar{p}_1)}{\partial \bar{p}} \; = \; \frac{1}{D \, E} \left[\, c_A \, E^2 - \left(\hat{E} + c_N \, E^2 \right) \right] \; = \; - \frac{1}{D \, E} \left[\, (c_N - c_A) \; E^2 + \hat{E} \, \right]$$

$$< \; 0 \;\; \text{if} \;\; c_N - c_A \; > \; - \frac{\hat{E}}{E^2} \;\; \Leftrightarrow \;\; \Phi_2 \; \equiv \; E^2 \, [\, c_N - c_A \,] + \hat{E} \; > \; 0 \, . \tag{211}$$

(211) reflects the facts that $E > 0$ (from (201)) and $D > 0$ (by assumption).

It remains to demonstrate that Φ_2 is as specified in (193). (201) and (210) imply:

$$\hat{E} \; = \; \psi_1 \, E + \psi_2 \, , \;\; \text{where} \tag{212}$$

$$\psi_1 \; \equiv \; a \left[\, b \, k + 2 \, b \, k^R + k \, k^R + b^2 \, \right] - a \left[\, b + k^R \, \right] [\, 2 \, b + k \,] + b \left[\, b + k^R \, \right] [\, a - c \,]$$
$$- \; \left[\, (b + k_N) \, (2 \, b + k) - b^2 \, \right] c_N, \;\; \text{and} \tag{213}$$

$$\psi_2 \; \equiv \; - \; \left[\, b \, (a - c_N) \, (b + k) + b \left(b + k_N + k^R \right) (a - c) \, \right] \left[\, b \, k + 2 \, b \, k^R + k \, k^R + b^2 \, \right]$$

$$+ \; b \, [\, b + k \,] \, [\, b + k_N \,] \, [\, (a - c_N) \, (2 \, b + k) - b \, (a - c) \,] \, . \tag{214}$$

(213) implies:

$$\psi_1 \; = \; a \left[\, b \, k + 2 \, b \, k^R + k \, k^R + b^2 - \left(b + k^R \right) (2 \, b + k) + b \left(b + k^R \right) \right]$$

$$- \; b \left[\, b + k^R \, \right] c - \left[\, (b + k_N) \, (2 \, b + k) - b^2 \, \right] c_N$$

$$= \; a \left[\, b \, k + 2 \, b \, k^R + k \, k^R + b^2 - 2 \, b^2 - b \, k - 2 \, b \, k^R - k \, k^R + b^2 + b \, k^R \right]$$

$$- \; b \left[\, b + k^R \, \right] c - \left[\, (b + k_N) \, (2 \, b + k) - b^2 \, \right] c_N$$

$$= \; a \, b \, k^R - b \left[\, b + k^R \, \right] c - \left[\, (b + k_N) \, (2 \, b + k) - b^2 \, \right] c_N \, . \tag{215}$$

(214) implies:

$$\psi_2 \; = \; - \left\{ \, b \, [\, b + k \,] \, [\, a - c_N \,] + b \left[\, b + k_N + k^R \, \right] [\, a - c \,] \, \right\} \left[\, b \, k + 2 \, b \, k^R + k \, k^R + b^2 \, \right]$$

$$+ \;\; b \, [\, b + k \,] \, [\, b + k_N \,] \, [\, 2 \, b + k \,] \, [\, a - c_N \,] - b^2 \, [\, b + k \,] \, [\, b + k_N \,] \, [\, a - c \,]$$

$$
\begin{aligned}
&= \; [\, a - c_N \,] \left\{ b\, [\, b + k \,]\, [\, b + k_N \,]\, [\, 2\, b + k \,] - b\, [\, b + k \,] \left[\, b\, k + 2\, b\, k^R + k\, k^R + b^2 \,\right] \right\} \\
&\quad - [\, a - c \,] \left\{ b^2\, [\, b + k \,]\, [\, b + k_N \,] + b \left[\, b + k_N + k^R \,\right] \left[\, b\, k + 2\, b\, k^R + k\, k^R + b^2 \,\right] \right\} \\
&= \; [\, a - c_N \,]\, b\, [\, b + k \,] \left\{ [\, b + k_N \,]\, [\, 2\, b + k \,] - \left[\, b\, k + 2\, b\, k^R + k\, k^R + b^2 \,\right] \right\} \\
&\quad - \; b\, [\, a - c \,] \left\{ b\, [\, b + k \,]\, [\, b + k_N \,] + \left[\, b + k_N + k^R \,\right] \left[\, b\, k + 2\, b\, k^R + k\, k^R + b^2 \,\right] \right\}. \qquad (216)
\end{aligned}
$$

The coefficient on $[\, a - c_N \,]\, b\, [\, b + k \,]$ in (216) is:

$$
\begin{aligned}
&2\, b^2 + b\, k + 2\, b\, k_N + k\, k_N - b\, k - 2\, b\, k^R - k\, k^R - b^2 \\
&\quad = \; b^2 + 2\, b\, k_N + k\, k_N - 2\, b\, k^R - k\, k^R \; = \; b^2 + [\, 2\, b + k \,] \left[\, k_N - k^R \,\right]. \qquad (217)
\end{aligned}
$$

The coefficient on $-\, b\, [\, a - c \,]$ in (216) is:

$$
\begin{aligned}
&b\, [\, b + k \,]\, [\, b + k_N \,] + [\, b + k_N \,] \left[\, b\, k + 2\, b\, k^R + k\, k^R + b^2 \,\right] \\
&\qquad + \; k^R \left[\, b\, k + 2\, b\, k^R + k\, k^R + b^2 \,\right] \\
&= \; [\, b + k_N \,] \left[\, b^2 + b\, k + b\, k + 2\, b\, k^R + k\, k^R + b^2 \,\right] + k^R \left[\, b\, k + 2\, b\, k^R + k\, k^R + b^2 \,\right] \\
&= \; [\, b + k_N \,] \left[\, 2\, b^2 + 2\, b\, k + 2\, b\, k^R + k\, k^R \,\right] + k^R \left[\, b\, k + 2\, b\, k^R + k\, k^R + b^2 \,\right] \\
&= \; [\, b + k_N \,] \left[\, 2\, b^2 + 2\, b\, k \,\right] + k^R \left[\, (b + k_N)\, (2\, b + k) + b\, k + 2\, b\, k^R + k\, k^R + b^2 \,\right] \\
&= \; 2\, b\, [\, b + k \,]\, [\, b + k_N \,] + k^R \left[\, 2\, b^2 + b\, k + 2\, b\, k_N + k\, k_N + b\, k + 2\, b\, k^R + k\, k^R + b^2 \,\right] \\
&= \; 2\, b\, [\, b + k \,]\, [\, b + k_N \,] + k^R \left[\, 3\, b^2 + 2\, b\, k + 2\, b\, k_N + k\, k_N + 2\, b\, k^R + k\, k^R \,\right] \\
&= \; 2\, b\, [\, b + k \,]\, [\, b + k_N \,] + k^R \left[\, 3\, b^2 + 2\, b\, k + \left(k_N + k^R \right) (2\, b + k) \,\right] \\
&= \; 2\, b\, [\, b + k \,]\, [\, b + k_N \,] + k^R\, E\,. \qquad (218)
\end{aligned}
$$

The last equality in (218) reflects (201).

(212) and (215) – (218) imply:

$$
\begin{aligned}
\hat{E} \; &= \; E \left\{ [\, a - c \,]\, b\, k^R - b^2 c - \left[\, (b + k_N)\, (2\, b + k) - b^2 \,\right] c_N \right\} \\
&\qquad + \left\{ b^2 + [\, 2\, b + k \,] \left[\, k_N - k^R \,\right] \right\} b\, [\, b + k \,]\, [\, a - c_N \,] \\
&\qquad - \left\{ 2\, b\, [\, b + k \,]\, [\, b + k_N \,] + k^R\, E \right\} b\, [\, a - c \,] \\
&= \; -\; E \left\{ b^2 c + \left[\, (b + k_N)\, (2\, b + k) - b^2 \,\right] c_N \right\} \\
&\qquad + b\, [\, b + k \,] \left\{ b^2 + [\, 2\, b + k \,] \left[\, k_N - k^R \,\right] \right\} [\, a - c_N \,] \\
&\qquad - 2\, b^2\, [\, b + k \,]\, [\, b + k_N \,]\, [\, a - c \,]\,. \qquad (219)
\end{aligned}
$$

(201) and (219) imply:

$$\Phi_2 \equiv E^2 [\, c_N - c_A \,] + \widehat{E}$$

$$= \Big\{ b [\, 3\, b + 2\, k \,] + [\, 2\, b + k \,] \big[\, k_N + k^R \,\big] \Big\}^2 [\, c_N - c_A \,]$$

$$- \Big\{ b [\, 3\, b + 2\, k \,] + [\, 2\, b + k \,] \big[\, k_N + k^R \,\big] \Big\}$$

$$\cdot \Big\{ b^2 c + \big[\, (b + k_N)\, (2\, b + k) - b^2 \,\big]\, c_N \Big\}$$

$$+ \; b [\, b + k \,] \Big\{ b^2 + [\, 2\, b + k \,] \big[\, k_N - k^R \,\big] \Big\} [\, a - c_N \,] - 2\, b^2 [\, b + k \,] [\, b + k_N \,] [\, a - c \,] \, . \quad (220)$$

Observe that:

$$[\, b + k_N \,] [\, 2\, b + k \,] - b^2 \; = \; b^2 + b\, k + k_N [\, 2\, b + k \,] \; = \; b [\, b + k \,] + [\, 2\, b + k \,]\, k_N . \qquad (221)$$

(220) and (221) imply:

$$\Phi_2 = \Big\{ b [\, 3\, b + 2\, k \,] + [\, 2\, b + k \,] \big[\, k_N + k^R \,\big] \Big\}^2 [\, c_N - c_A \,]$$

$$- \Big\{ b [\, 3\, b + 2\, k \,] + [\, 2\, b + k \,] \big[\, k_N + k^R \,\big] \Big\}$$

$$\cdot \Big\{ b^2 c + [\, b\, (b + k) + (2\, b + k)\, k_N \,]\, c_N \Big\}$$

$$+ \; b [\, b + k \,] \Big\{ b^2 + [\, 2\, b + k \,] \big[\, k_N - k^R \,\big] \Big\} [\, a - c_N \,]$$

$$- \; 2\, b^2 [\, b + k \,] [\, b + k_N \,] [\, a - c \,]$$

$$= \Big\{ b [\, 3\, b + 2\, k \,] + [\, 2\, b + k \,] \big[\, k_N + k^R \,\big] \Big\}$$

$$\cdot \Bigg[\Big\{ b [\, 3\, b + 2\, k \,] + [\, 2\, b + k \,] \big[\, k_N + k^R \,\big] \Big\} [\, c_N - c_A \,]$$

$$- \; b^2 c - b [\, b + k \,]\, c_N - [\, 2\, b + k \,]\, k_N\, c_N \Bigg]$$

$$+ \, b [\, b + k \,] \; \{ b^2 [\, a - c_N \,] + [\, 2\, b + k \,] \big[\, k_N - k^R \,\big] [\, a - c_N \,]$$

$$- \; 2\, b [\, b + k_N \,] [\, a - c \,] \}$$

$$= \Big\{ b [\, 3\, b + 2\, k \,] + [\, 2\, b + k \,] \big[\, k_N + k^R \,\big] \Big\}$$

$$\cdot \Bigg[\Big\{ b [\, 3\, b + 2\, k \,] + [\, 2\, b + k \,] \big[\, k_N + k^R \,\big] - b [\, b + k \,] - [\, 2\, b + k \,]\, k_N \Big\} \, c_N$$

$$- \, b^2 c - \Big\{ b [\, 3\, b + 2\, k \,] + [\, 2\, b + k \,] \big[\, k_N + k^R \,\big] \Big\} \, c_A \Bigg]$$

$$+ \, b [\, b + k \,] \Bigg[\Big\{ b^2 + [\, 2\, b + k \,] \big[\, k_N - k^R \,\big] - 2\, b [\, b + k_N \,] \Big\} \, a$$

$$- \left\{ b^2 + [\,2\,b + k\,]\left[\,k_N - k^R\,\right] \right\} c_N + 2\,b\,[\,b + k_N\,]\,c \Bigg] . \tag{222}$$

Observe that:

$$\begin{aligned} & b\,[\,3\,b + 2\,k\,] + [\,2\,b + k\,]\left[\,k_N + k^R\,\right] - b\,[\,b + k\,] - [\,2\,b + k\,]\,k_N \\ & \quad = b\,[\,3\,b + 2\,k - b - k\,] + [\,2\,b + k\,]\,k^R = [\,2\,b + k\,]\left[\,b + k^R\,\right] . \end{aligned} \tag{223}$$

Further observe that:

$$\begin{aligned} & b^2 + [\,2\,b + k\,]\left[\,k_N - k^R\,\right] - 2\,b\,[\,b + k_N\,] \\ & \quad = b^2 + [\,2\,b + k - 2\,b\,]\,k_N - [\,2\,b + k\,]\,k^R - 2\,b^2 \\ & \quad = -\,b^2 + k\,k_N - [\,2\,b + k\,]\,k^R = -\left[\,b^2 - k\,k_N + (2\,b + k)\,k^R\,\right] . \end{aligned} \tag{224}$$

(222) – (224) imply:

$$\begin{aligned} & E^2\,[\,c_N - c_A\,] + \hat{E} \\ & \quad = \left\{ b\,[\,3\,b + 2\,k\,] + [\,2\,b + k\,]\left[\,k_N + k^R\,\right] \right\} \\ & \qquad \cdot \Bigg\{ [\,2\,b + k\,]\left[\,b + k^R\,\right] c_N - b^2\,c \\ & \qquad\qquad - \left\{ b\,[\,3\,b + 2\,k\,] + [\,2\,b + k\,]\left[\,k_N + k^R\,\right] \right\} c_A \Bigg\} \\ & \qquad - b\,[\,b + k\,] \Bigg\{ \left[\,b^2 - k\,k_N + (2\,b + k)\,k^R\,\right] a \\ & \qquad\qquad + \left[\,b^2 + (2\,b + k)\,\left(k_N - k^R\right)\,\right] c_N - 2\,b\,[\,b + k_N\,]\,c \Bigg\} . \end{aligned} \tag{225}$$

The coefficient on c_N in (225) is:

$$\begin{aligned} & b\,[\,2\,b + k\,][\,3\,b + 2\,k\,]\left[\,b + k^R\,\right] + [\,2\,b + k\,]^2\left[\,b + k^R\,\right]\left[\,k_N + k^R\,\right] \\ & \quad - b^3\,[\,b + k\,] - b\,[\,b + k\,][\,2\,b + k\,]\left[\,k_N - k^R\,\right] \\ & \quad = k^R \left\{ b\,[\,2\,b + k\,][\,3\,b + 2\,k\,] + [\,2\,b + k\,]^2\left[\,b + k^R\,\right] + b\,[\,b + k\,][\,2\,b + k\,] \right\} \\ & \qquad + k_N \left\{ [\,2\,b + k\,]^2\left[\,b + k^R\,\right] - b\,[\,b + k\,][\,2\,b + k\,] \right\} \\ & \qquad + b^2\,[\,2\,b + k\,][\,3\,b + 2\,k\,] - b^3\,[\,b + k\,] \\ & \quad = k^R\,[\,2\,b + k\,] \left\{ b\,[\,3\,b + 2\,k\,] + [\,2\,b + k\,]\left[\,b + k^R\,\right] + b\,[\,b + k\,] \right\} \\ & \qquad + k_N\,[\,2\,b + k\,] \left\{ [\,2\,b + k\,]\left[\,b + k^R\,\right] - b\,[\,b + k\,] \right\} \end{aligned}$$

$$+\ b^2 \{ [2b+k][3b+2k] - b[b+k] \}$$

$$= \ k^R [2b+k] \Big\{ k^R [2b+k] + b[3b+2k+2b+k+b+k] \Big\}$$

$$+\ k_N [2b+k] \Big\{ k^R [2b+k] + b[2b+k-b-k] \Big\}$$

$$+\ b^2 \Big[6b^2 + 7bk + 2k^2 - b^2 - bk \Big]$$

$$= \ k^R [2b+k] \Big[k^R (2b+k) + 2b(3b+2k) \Big]$$

$$+\ \ k_N [2b+k] \Big[k^R (2b+k) + b^2 \Big] + \ b^2 \Big[5b^2 + 6bk + 2k^2 \Big] . \qquad (226)$$

The coefficient on c in (225) is:

$$2b^2 [b+k][b+k_N] - b^2 \Big\{ b[3b+2k] + [2b+k] \Big[k_N + k^R \Big] \Big\}$$

$$= \ b^2 \Big\{ 2[b+k][b+k_N] - b[3b+2k] - [2b+k] \Big[k_N + k^R \Big] \Big\}$$

$$= \ b^2 \Big\{ 2 \Big[b^2 + bk + bk_N + kk_N \Big] - 3b^2 - 2bk - 2bk_N - kk_N - [2b+k] k^R \Big\}$$

$$= \ b^2 \Big\{ -b^2 + kk_N - [2b+k] k^R \Big\} \ = \ -b^2 \Big[b^2 - kk_N + (2b+k) k^R \Big] . \qquad (227)$$

(201) implies that the coefficient on c_A in (225) is $-E^2$. Therefore, (201) and (225) – (227) imply:

$$\Phi_2 \ = \ \Big\{ k^R [2b+k] \Big[k^R (2b+k) + 2b(3b+2k) \Big]$$

$$+\ \ k_N [2b+k] \Big[k^R (2b+k) + b^2 \Big] + b^2 \Big[5b^2 + 6bk + 2k^2 \Big] \Big\} \ c_N$$

$$-\ b^2 \Big[b^2 - kk_N + (2b+k) k^R \Big] \ c - \ E^2 c_A$$

$$-\ b[b+k] \Big[b^2 - kk_N + (2b+k) k^R \Big] \ a$$

$$= \ \Big\{ k^R [2b+k] \Big[k^R (2b+k) + 2b(3b+2k) \Big]$$

$$+\ \ k_N [2b+k] \Big[k^R (2b+k) + b^2 \Big] + b^2 \Big[5b^2 + 6bk + 2k^2 \Big] \Big\} \ c_N$$

$$-\ b \Big[b^2 - kk_N + (2b+k) k^R \Big] [a(b+k) + bc]$$

$$-\ \ \Big\{ b[3b+2k] + [2b+k] \Big[k_N + k^R \Big] \Big\}^2 c_A . \qquad (228)$$

It remains to prove that $\bar{p}_{V_2m} < \bar{p}_2$, which is established by demonstrating that $\left.\frac{\partial^- V(\bar{p})}{\partial \bar{p}}\right|_{\bar{p}=\bar{p}_2} > 0$. Define $V_2(\bar{p}) \equiv \bar{p}\, q_A(\cdot) + P(Q(\cdot))\, q_N(\cdot)$ for $\bar{p} \in (\bar{p}_1, \bar{p}_2)$. Because $P(Q) = a - bQ$:

$$\frac{\partial^- V_2(\bar{p}_2)}{\partial \bar{p}} \ = \ q_A + \bar{p}_2 \frac{\partial q_A}{\partial \bar{p}} + P(Q) \frac{\partial q_N}{\partial \bar{p}} - b\, q_N \frac{\partial Q}{\partial \bar{p}} \qquad (229)$$

where q_A, q_N, and Q are as specified in Lemma A2, evaluated at $\bar{p} = \bar{p}_2$. Because $\bar{p}_2 = P(Q)$,

(229) implies:

$$\frac{\partial^- V_2(\bar{p}_2)}{\partial \bar{p}} \;=\; q_A + \bar{p}_2 \left[\frac{\partial q_A}{\partial \bar{p}} + \frac{\partial q_N}{\partial \bar{p}} \right] - b\, q_N \frac{\partial Q}{\partial \bar{p}} \,. \tag{230}$$

(68) implies:

$$\begin{aligned}
\bar{p}_2 \;&=\; \left[b + k^R \right] Q^R + c_N + k_N\, q_N - b\, q_A \\
&=\; \left[b + k^R \right] q_A + \left[b + k^R \right] q_N + c_N + k_N\, q_N - b\, q_A \\
&=\; k^R\, q_A + \left[b + k_N + k^R \right] q_N + c_N \,.
\end{aligned} \tag{231}$$

(230) and (231) imply:

$$\begin{aligned}
\frac{\partial^- V_2(\bar{p}_2)}{\partial \bar{p}} \;&=\; q_A - b\, q_N \frac{\partial Q}{\partial \bar{p}} + \left[k^R\, q_A + \left(b + k_N + k^R \right)\, q_N + c_N \right] \left[\frac{\partial q_A}{\partial \bar{p}} + \frac{\partial q_N}{\partial \bar{p}} \right] \\
&=\; q_A + \left[k^R\, q_A + \left(k_N + k^R \right)\, q_N + c_N \right] \left[\frac{\partial q_A}{\partial \bar{p}} + \frac{\partial q_N}{\partial \bar{p}} \right] \\
&\qquad + b\, q_N \left[\frac{\partial q_A}{\partial \bar{p}} + \frac{\partial q_N}{\partial \bar{p}} \right] - b\, q_N \frac{\partial Q}{\partial \bar{p}} \\
&=\; q_A + \left[k^R\, q_A + \left(k_N + k^R \right)\, q_N + c_N \right] \left[\frac{\partial q_A}{\partial \bar{p}} + \frac{\partial q_N}{\partial \bar{p}} \right] \\
&\qquad + b\, q_N \left[\frac{\partial q_A}{\partial \bar{p}} + \frac{\partial q_N}{\partial \bar{p}} - \frac{\partial Q}{\partial \bar{p}} \right] \\
&=\; q_A + \left[k^R\, q_A + \left(k_N + k^R \right)\, q_N + c_N \right] \left[\frac{\partial q_A}{\partial \bar{p}} + \frac{\partial q_N}{\partial \bar{p}} \right] - b\, q_N \frac{\partial q}{\partial \bar{p}} \;>\; 0 \,.
\end{aligned} \tag{232}$$

The inequality holds here because $\frac{\partial q_A}{\partial \bar{p}} + \frac{\partial q_N}{\partial \bar{p}} = \frac{\partial Q^R}{\partial \bar{p}} > 0$ (from (22)) and $\frac{\partial q}{\partial \bar{p}} < 0$ (from (23)). ■

Lemma 5. *For $\bar{p} \in (\bar{p}_1, \bar{p}_2)$: (i) $S(\bar{p})$ is a strictly concave function of $\bar{p}$; (ii) $\frac{\partial S(\overline{p})}{\partial \overline{p}} \gtreqless 0 \Leftrightarrow \overline{p} \lesseqgtr \overline{p}_{S_2 M}$ where $\overline{p}_{S_2 M} \in (\overline{p}_1, \overline{p}_2]$; and (iii) $\overline{p}_{S_2 M} > \overline{p}_{V_2 m}$.*

Proof. As in (173), define:

$$\widetilde{S}_2(\bar{p}) \;\equiv\; a\; Q_2(\bar{p}) - \frac{b}{2}\; Q_2(\bar{p})^2 - q_{A2}(\bar{p})\, \bar{p} - \left[q_2(\bar{p}) + q_{N2}(\bar{p}) \right] P(Q_2(\bar{p})) \tag{233}$$

where $q_{A2}(\bar{p})$, $q_{N2}(\bar{p})$, $q_2(\bar{p})$, and $Q_2(\bar{p})$ are as defined in (20), (21), (23), and (24), respectively. Observe that $\widetilde{S}_2(\bar{p}) = S(\bar{p})$ for $\bar{p} \in [\bar{p}_1, \bar{p}_2]$.

(233) implies that because $P(Q_2) = a - b\, Q_2$ and $Q_2 = q_{A2} + q_{N2} + q_2$:

$$\frac{\partial \widetilde{S}_2(\bar{p})}{\partial \bar{p}} = a \frac{\partial Q_2}{\partial \bar{p}} - b\, Q_2 \frac{\partial Q_2}{\partial \bar{p}} - q_{A2} - \bar{p} \frac{\partial q_{A2}}{\partial \bar{p}} - P(Q_2) \left[\frac{\partial q_{N2}}{\partial \bar{p}} + \frac{\partial q_2}{\partial \bar{p}} \right] + b \frac{\partial Q_2}{\partial \bar{p}} [\, q_{N2} + q_2 \,]$$

$$= P(Q_2) \left[\frac{\partial Q_2}{\partial \bar{p}} - \frac{\partial q_{N2}}{\partial \bar{p}} - \frac{\partial q_2}{\partial \bar{p}} \right] - \bar{p} \frac{\partial q_{A2}}{\partial \bar{p}} + b \frac{\partial Q_2}{\partial \bar{p}} [\, q_{N2} + q_2 \,] - q_{A2}$$

$$= [\, P(Q_2) - \bar{p} \,] \frac{\partial q_{A2}}{\partial \bar{p}} + b \frac{\partial Q_2}{\partial \bar{p}} [\, q_{N2} + q_2 \,] - q_{A2} \,. \tag{234}$$

(196) and (234) imply that because $P(Q_2) = a - b\, Q_2$:

$$\frac{\partial^2 \widetilde{S}_2(\bar{p})}{\partial\, (\bar{p})^2} = \left[- b \frac{\partial Q_2}{\partial \bar{p}} - 1 \right] \frac{\partial q_{A2}}{\partial \bar{p}} + b \frac{\partial Q_2}{\partial \bar{p}} \left[\frac{\partial q_{N2}}{\partial \bar{p}} + \frac{\partial q_2}{\partial \bar{p}} \right] - \frac{\partial q_{A2}}{\partial \bar{p}} < 0 \,. \tag{235}$$

The inequality in (235) holds because $D > 0$ by assumption, so $\frac{\partial q_{A2}}{\partial \bar{p}} > 0$ from (20), $\frac{\partial Q_2}{\partial \bar{p}} > 0$ from (24), $\frac{\partial q_{N2}}{\partial \bar{p}} < 0$ from (21), and $\frac{\partial q_2}{\partial \bar{p}} < 0$ from (23).

$\bar{p}_{S_2 M} \equiv \arg\max_{\bar{p}} \{ \widetilde{S}_2(\bar{p}) \}$ is unique and is determined by:

$$\frac{\partial \widetilde{S}_2(\bar{p}_{S_2 M})}{\partial \bar{p}} \equiv \left. \frac{\partial \widetilde{S}_2(\bar{p})}{\partial \bar{p}} \right|_{\bar{p} = \bar{p}_{S_2 M}} = 0 \,. \tag{236}$$

This is the case because (2), (20) – (24), and (234) imply that $\frac{\partial \widetilde{S}_2(\bar{p})}{\partial \bar{p}}$ is a linear function of $\bar{p}$. Therefore, $\widetilde{S}_2(\bar{p})$ is a quadratic function of $\bar{p}$. Consequently, (235) implies that $\widetilde{S}_2(\bar{p})$ has a unique maximum that is determined by (236).

To prove that $\bar{p}_{S_2 M} > \bar{p}_{V_2 m}$, let:

$$H(\bar{p}) \equiv a\, Q_2 - \frac{b}{2} Q_2^2 - [\, a - b\, Q_2 \,]\, q_2 \tag{237}$$

$$\Rightarrow \quad \frac{\partial H(\bar{p})}{\partial \bar{p}} \equiv [\, a - b\, Q_2 \,] \frac{\partial Q_2}{\partial \bar{p}} - [\, a - b\, Q_2 \,] \frac{\partial q_2}{\partial \bar{p}} + b \frac{\partial Q_2}{\partial \bar{p}}\, q_2 \tag{238}$$

where q_2 and Q_2 are defined in (23) and (24). Differentiating (238) provides:

$$\frac{\partial^2 H(\bar{p})}{(\partial \bar{p})^2} \equiv - b \left(\frac{\partial Q_2}{\partial \bar{p}} \right)^2 + 2\, b \frac{\partial Q_2}{\partial \bar{p}} \frac{\partial q_2}{\partial \bar{p}} < 0 \,. \tag{239}$$

The inequality in (239) holds because $\frac{\partial Q_2}{\partial \bar{p}} > 0$ and $\frac{\partial q_2}{\partial \bar{p}} < 0$, from (23) and (24). (238) implies:

$$\frac{\partial H(\bar{p}_2)}{\partial \bar{p}} \equiv \left. \frac{\partial H(\bar{p})}{\partial \bar{p}} \right|_{\bar{p} = \bar{p}_2} = \bar{p}_2 \frac{\partial Q_2}{\partial \bar{p}} - \bar{p}_2 \frac{\partial q_2}{\partial \bar{p}} + b \frac{\partial Q_2}{\partial \bar{p}}\, q_2(\bar{p}_2) > 0 \,. \tag{240}$$

The inequality in (240) holds because $\frac{\partial Q_2}{\partial \bar{p}} > 0$ and $\frac{\partial q_2}{\partial \bar{p}} < 0$, from (23) and (24). The concavity of $H(\bar{p})$ established in (239), along with (240), imply:

$$\frac{\partial H(\bar{p})}{\partial \bar{p}} \;>\; 0 \text{ for all } \bar{p} \;<\; \bar{p}_2 \quad \Rightarrow \quad \frac{\partial H(\bar{p}_{V_2m})}{\partial \bar{p}} \;>\; 0\,. \tag{241}$$

The implication in (241) holds because $\bar{p}_{V_2m} < \bar{p}_2$, from Lemma 4.

(195) and (236) imply:

$$\frac{\partial \widetilde{V}_2(\bar{p}_{V_2m})}{\partial \bar{p}} \;=\; [\, a - b\, Q_2(\cdot)\,]\; \frac{\partial q_{N2}(\cdot)}{\partial \bar{p}} - b\; \frac{\partial Q_2(\cdot)}{\partial \bar{p}}\; q_{N2}(\cdot)$$

$$+\; q_{A2}(\cdot) + \bar{p}_{V_2m}\; \frac{\partial q_{A2}(\cdot)}{\partial \bar{p}} \;=\; 0 \tag{242}$$

where $q_{A2}(\cdot)$, $q_{N2}(\cdot)$, and $Q_2(\cdot)$ are defined in (20), (21), and (24), and evaluated at $\bar{p}_{V_2m}$.

(234) implies:

$$\frac{\partial \widetilde{S}_2(\bar{p})}{\partial \bar{p}} \;=\; [\, a - b\, Q_2\,]\; \frac{\partial Q_2}{\partial \bar{p}} - [\, a - b\, Q_2\,]\; \frac{\partial q_2}{\partial \bar{p}} + b\; \frac{\partial Q_2}{\partial \bar{p}}\; q_2$$

$$-\; [\, a - b\, Q_2\,]\; \frac{\partial q_{N2}}{\partial \bar{p}} + b\; \frac{\partial Q_2}{\partial \bar{p}}\; q_{N2} - q_{A2} - \bar{p}\; \frac{\partial q_{A2}}{\partial \bar{p}} \tag{243}$$

where q_{A2}, q_{N2}, q_2, and Q_2 are defined in (20), (21), (23), and (24). (243) implies:

$$\begin{aligned}
\frac{\partial \widetilde{S}_2(\bar{p}_{V_2m})}{\partial \bar{p}} \;&=\; [\, a - b\, Q_2(\bar{p}_{V_2m})\,]\; \frac{\partial Q_2}{\partial \bar{p}} - [\, a - b\, Q_2(\bar{p}_{V_2m})\,]\; \frac{\partial q_2}{\partial \bar{p}} + b\; \frac{\partial Q_2}{\partial \bar{p}}\; q_2(\bar{p}_{V_2m}) \\
&\quad -\; [\, a - b\, Q_2(\bar{p}_{V_2m})\,]\; \frac{\partial q_{N2}}{\partial \bar{p}} + b\; \frac{\partial Q_2}{\partial \bar{p}}\; q_{N2}(\bar{p}_{V_2m}) - q_{A2}(\bar{p}_{V_2m}) - \bar{p}_{V_2m}\; \frac{\partial q_{A2}}{\partial \bar{p}} \\
&=\; [\, a - b\, Q_2(\bar{p}_{V_2m})\,]\; \frac{\partial Q_2}{\partial \bar{p}} - [\, a - b\, Q_2(\bar{p}_{V_2m})\,]\; \frac{\partial q_2}{\partial \bar{p}} + b\; \frac{\partial Q_2}{\partial \bar{p}}\; q_2(\bar{p}_{V_2m}) \\
&=\; \frac{\partial H(\bar{p}_{V_2m})}{\partial \bar{p}} \;>\; 0\,.
\end{aligned} \tag{244}$$

The last equality in (244) reflects (242). The inequality in (244) reflects (241).

(235) implies that $\widetilde{S}_2(\bar{p})$ is a strictly concave function of $\bar{p}$. Therefore, $\bar{p}_{V_2m} < \bar{p}_{S_2M}$ because: (i) $\frac{\partial \widetilde{S}_2(\bar{p}_{S_2M})}{\partial \bar{p}} = 0$ from (236); and (ii) $\frac{\partial \widetilde{S}_2(\bar{p}_{V_2m})}{\partial \bar{p}} > 0$, from (244).

To prove that $\bar{p}_{S_2M} > \bar{p}_1$, it suffices to establish that $\frac{\partial^+ S_2(\bar{p}_1)}{\partial \bar{p}} \equiv \left.\frac{\partial^+ S_2(\bar{p}_1)}{\partial \bar{p}}\right|_{\bar{p}=\bar{p}_1} > 0$. Lemma A1 implies that $q_A = 0$ when $\bar{p} = \bar{p}_1$. Therefore, (173) implies:

$$\begin{aligned}
\frac{\partial^+ \widetilde{S}_2(\bar{p}_1)}{\partial \bar{p}} \;&=\; [\, a - b\, Q\,]\, \frac{\partial Q}{\partial \bar{p}} - \bar{p}_1\; \frac{\partial q_A}{\partial \bar{p}} - P(Q) \left[\frac{\partial q_N}{\partial \bar{p}} + \frac{\partial q}{\partial \bar{p}} \right] + b\, [\, q_N + q\,]\, \frac{\partial Q}{\partial \bar{p}} \\
&=\; P(Q)\; \frac{\partial Q}{\partial \bar{p}} - \bar{p}_1\; \frac{\partial q_A}{\partial \bar{p}} - P(Q) \left[\frac{\partial q_N}{\partial \bar{p}} + \frac{\partial q}{\partial \bar{p}} \right] + b\, [\, q_N + q\,]\, \frac{\partial Q}{\partial \bar{p}}
\end{aligned}$$

$$= \left[P(Q) - \bar{p}_1 \right] \frac{\partial q_A}{\partial \bar{p}} + b \left[q_N + q \right] \frac{\partial Q}{\partial \bar{p}} \; > \; 0 \, . \tag{245}$$

The inequality in (245) holds because $D > 0$ by assumption, so $\frac{\partial q_A}{\partial \bar{p}} > 0$ from (20), $\frac{\partial Q}{\partial \bar{p}} > 0$ from (24), and $P(Q) > \bar{p}_1$ when $\bar{p} \in (\bar{p}_1, \bar{p}_2)$. ■

Proposition 6. *$\bar{p}^* > \bar{p}_1$ if $\Phi_2 \geq 0$. $\bar{p}^* = \bar{p}_1$ if $\Phi_2 < 0$ and d is sufficiently large.*

Proof. The first conclusion in the Proposition holds because (172) implies that when if $\Phi_2 \geq 0$:

$$\frac{\partial^+ W_2(\bar{p}_1)}{\partial \bar{p}} \; \equiv \; \left. \frac{\partial^+ W_2(\bar{p})}{\partial \bar{p}} \right|_{\bar{p} = \bar{p}_1} \; = \; \frac{\partial^+ S_2(\bar{p}_1)}{\partial \bar{p}} - d \, \frac{\partial^+ V_2(\bar{p}_1)}{\partial \bar{p}} \; > \; 0 \, . \tag{246}$$

The inequality in (246) holds because when $\Phi_2 \geq 0$: (i) $\frac{\partial^+ V_2(\bar{p}_1)}{\partial \bar{p}} < 0$ from (211); and (ii) $\frac{\partial^+ S_2(\bar{p}_1)}{\partial \bar{p}} > 0$ from (245).

The second conclusion in the Proposition holds if $V(\bar{p}_1) < V(\bar{p})$ for all $\bar{p} > \bar{p}_1$ when d is sufficiently large and $\Phi_2 < 0$. (211) and (220) imply:

$$\left. \frac{\partial^+ V(\bar{p})}{\partial \bar{p}} \right|_{\bar{p} = \bar{p}_1} \; > \; 0 \text{ when } \Phi_2 \; < \; 0 \, . \tag{247}$$

$V(\bar{p})$ is a strictly convex function of $\bar{p}$ for $\bar{p} \in (\bar{p}_1, \bar{p}_2)$, from Lemma 4. Therefore, (247) implies that $V(\bar{p})$ is a strictly increasing function of $\bar{p}$ for $\bar{p} \in [\, \bar{p}_1, \bar{p}_2 \,]$ under the maintained conditions. Consequently:

$$V(\bar{p}_1) \; < \; V(\bar{p}) \text{ for all } \bar{p} \; \in \; (\bar{p}_1, \bar{p}_2 \,] \, . \tag{248}$$

Lemma 2 implies that under the maintained conditions:

$$V(\bar{p}_1) \; < \; V(\bar{p}_3) \, . \tag{249}$$

(109) implies that $V(\bar{p})$ is a strictly concave function of $\bar{p}$ for $\bar{p} \in (\, \bar{p}_2, \bar{p}_3 \,)$. Therefore, (248) and (249) imply:

$$V(\bar{p}) \; > \; V(\bar{p}_1) \text{ for all } \bar{p} \; \in \; (\bar{p}_2, \bar{p}_3 \,] \, . \tag{250}$$

The conclusion follows from (248), (250), and Proposition 5. ■

Proposition 7. *$\bar{p}^* \in [\, \overline{p}_{V_2 m}, \overline{p}_{S_2 M} \,]$. Furthermore: (i) $\bar{p}^* < \overline{p}_{S_2 M}$ when $\overline{p}_{S_2 M} < \overline{p}_2$ and $d > 0$; (ii) $\bar{p}^* > \overline{p}_{V_2 m}$ when $\overline{p}_{V_2 m} > \overline{p}_1$; (iii) $\bar{p}^* \to \overline{p}_{S_2 M}$ as $d \to 0$; and (iv) $\bar{p}^* \to \overline{p}_{V_2 m}$ as $d \to \infty$.*

Proof. To prove that $\bar{p}^* \leq \overline{p}_{S_2 M}$, suppose that $\bar{p}^* > \overline{p}_{S_2 M}$. $\widetilde{S}_2(\bar{p})$ is a strictly concave function of $\bar{p}$, from Lemma 5. Therefore, because $\bar{p}^* > \overline{p}_{S_2 M}$, (236) implies:

$$\frac{\partial \widetilde{S}_2(\bar{p}^*)}{\partial \bar{p}} \; < \; \frac{\partial \widetilde{S}_2(\overline{p}_{S_2 M})}{\partial \bar{p}} \; = \; 0 \, . \tag{251}$$

$\widetilde{V}_2(\bar{p})$ is a strictly convex function of $\bar{p}$, from Lemma 4. Therefore, because $\bar{p}_{V_2m} < \bar{p}_{S_2M}$ from Lemma 5 and because $\bar{p}^* > \bar{p}_{S_2M}$ by assumption, (198) implies:

$$\frac{\partial \widetilde{V}_2(\bar{p}^*)}{\partial \bar{p}} > \frac{\partial \widetilde{V}_2(\bar{p}_{S_2M})}{\partial \bar{p}} > \frac{\partial \widetilde{V}_2(\bar{p}_{V_2m})}{\partial \bar{p}} = 0 . \tag{252}$$

(251) and (252) imply that R's revenue declines and consumer surplus increases as $\bar{p}$ declines below $\bar{p}^*$. Therefore, $\bar{p}^*$ is not the welfare-maximizing value of $\bar{p}$. Hence, by contradiction, $\bar{p}^* \leq \bar{p}_{S_2M}$.

To prove that $\bar{p}^* \geq \bar{p}_{V_2m}$, suppose that $\bar{p}^* < \bar{p}_{V_2m}$. $\widetilde{V}_2(\bar{p})$ is a strictly convex function of $\bar{p}$, from Lemma 4. Therefore, because $\bar{p}_{V_2m} < \bar{p}_{S_2M}$ from Lemma 5, (198) implies:

$$\frac{\partial \widetilde{V}_2(\bar{p}^*)}{\partial \bar{p}} < \frac{\partial \widetilde{V}_2(\bar{p}_{V_2m})}{\partial \bar{p}} = 0 . \tag{253}$$

$\widetilde{S}_2(\bar{p})$ is a strictly concave function of $\bar{p}$, from Lemma 5. Therefore, because $\bar{p}_{V_2m} < \bar{p}_{S_2M}$ from Lemma 5 and because $\bar{p}^* < \bar{p}_{V_2m}$ by assumption, (236) implies:

$$\frac{\partial \widetilde{S}_2(\bar{p}^*)}{\partial \bar{p}} > \frac{\partial \widetilde{S}_2(\bar{p}_{V_2m})}{\partial \bar{p}} > \frac{\partial \widetilde{S}_2(\bar{p}_{S_2M})}{\partial \bar{p}} = 0 . \tag{254}$$

(253) and (254) imply that R's revenue declines and consumer surplus increases as $\bar{p}$ increases above $\bar{p}^*$. Therefore, $\bar{p}^*$ is not the welfare-maximizing value of $\bar{p}$. Hence, by contradiction, $\bar{p}^* \geq \bar{p}_{V_2m}$.

To prove conclusion (i) in the Proposition, define $\widetilde{W}_2(\cdot) \equiv \widetilde{S}_2(\cdot) - d\,\widetilde{V}_2(\cdot)$ and observe that when $\overline{p}_{S_2M} < \overline{p}_2$ and $d > 0$:

$$\begin{aligned} \left.\frac{\partial \widetilde{W}_2(\bar{p})}{\partial \bar{p}}\right|_{\bar{p}=\overline{p}_{S_2M}} &= \frac{\partial \widetilde{S}_2(\overline{p}_{S_2M})}{\partial \bar{p}} - d\,\frac{\partial \widetilde{V}_2(\overline{p}_{S_2M})}{\partial \overline{p}} \\ &= -\,d\,\frac{\partial \widetilde{V}_2(\overline{p}_{S_2M})}{\partial \overline{p}} < -\,d\,\frac{\partial \widetilde{V}_2(\overline{p}_{V_2m})}{\partial \overline{p}} = 0 . \end{aligned} \tag{255}$$

The inequality in (255) holds because: (i) $\overline{p}_{S_2M} > \overline{p}_{V_2m}$, from Lemma 5; and (ii) $\widetilde{V}_2(\cdot)$ is a strictly convex function of $\overline{p}$, from Lemma 4. (255) implies that $\overline{p}_{S_2M} > \overline{p}^*$ because $\widetilde{W}_2(\cdot)$ is a strictly concave function of $\overline{p}$ (because $\widetilde{S}_2(\cdot)$ is a strictly concave function of $\overline{p}$ and $\widetilde{V}_2(\cdot)$ is a strictly convex function of $\overline{p}$).

To prove conclusion (ii) in the Proposition, observe that when $\overline{p}_{V_2m} > \overline{p}_1$:

$$\begin{aligned} \left.\frac{\partial \widetilde{W}_2(\bar{p})}{\partial \bar{p}}\right|_{\bar{p}=\overline{p}_{V_2m}} &= \frac{\partial \widetilde{S}_2(\overline{p}_{V_2m})}{\partial \bar{p}} - d\,\frac{\partial \widetilde{V}_2(\overline{p}_{V_2m})}{\partial \overline{p}} \\ &= \frac{\partial \widetilde{S}_2(\overline{p}_{V_2m})}{\partial \bar{p}} > \frac{\partial \widetilde{S}_2(\overline{p}_{S_2M})}{\partial \bar{p}} = 0 . \end{aligned} \tag{256}$$

The inequality in (255) holds because: (i) $\overline{p}_{S_2M} > \overline{p}_{V_2m}$, from Lemma 5; and (ii) $\widetilde{S}_2(\cdot)$ is a strictly concave function of $\bar{p}$, from Lemma 5. (256) implies that $\bar{p}^* > \bar{p}_{V_2 m}$ because $\widetilde{W}_2(\cdot)$ is a strictly concave function of $\bar{p}$.

Conclusions (iii) and (iv) in the Proposition follow immediately from (172) because $\bar{p}^* \in (\bar{p}_1, \bar{p}_2)$ is a non-increasing function of d. This is the case because (172) implies that when $\bar{p}^* \in (\bar{p}_1, \bar{p}_2)$:

$$\frac{\partial S(\bar{p}^*)}{\partial \bar{p}} - d\,\frac{\partial \widetilde{V}(\bar{p}^*)}{\partial \bar{p}} = 0 \;\Rightarrow\; \frac{\partial^2 \widetilde{S}(\bar{p}^*)}{\partial(\bar{p})^2}\,\frac{\partial \bar{p}^*}{\partial d} - \frac{\partial \widetilde{V}(\bar{p}^*)}{\partial \bar{p}} - d\,\frac{\partial^2 \widetilde{V}(\bar{p}^*)}{\partial(\bar{p})^2}\,\frac{\partial \bar{p}^*}{\partial d} = 0$$

$$\Rightarrow\; \frac{\partial \bar{p}^*}{\partial d} = \frac{\frac{\partial \widetilde{V}(\bar{p}^*)}{\partial \bar{p}}}{\frac{\partial^2 \widetilde{S}(\bar{p}^*)}{\partial(\bar{p})^2} - d\,\frac{\partial^2 \widetilde{V}(\bar{p}^*)}{\partial(\bar{p})^2}} \stackrel{s}{=} -\,\frac{\partial \widetilde{V}(\bar{p}^*)}{\partial \bar{p}}\,. \tag{257}$$

The last conclusion in (257) holds because Lemmas 4 and 5 imply that $\frac{\partial^2 \widetilde{S}(\bar{p}^*)}{\partial(\bar{p})^2} < 0$ and $\frac{\partial^2 \widetilde{V}(\bar{p}^*)}{\partial(\bar{p})^2} > 0$ when $\bar{p}^* \in (\bar{p}_1, \bar{p}_2)$.

It remains to prove that $\frac{\partial \widetilde{V}_2(\bar{p}^*)}{\partial \bar{p}} \geq 0$. To do so, suppose that $\frac{\partial \widetilde{V}_2(\bar{p}^*)}{\partial \bar{p}} < 0$. Then:

$$\bar{p}^* < \bar{p}_{V_2 m}\,. \tag{258}$$

(258) holds because: (i) $\widetilde{V}_2(\bar{p})$ is a strictly convex function of $\bar{p}$, from Lemma 4; and (ii) $\frac{\partial \widetilde{V}_2(\bar{p}_{V_2 m})}{\partial \bar{p}} = 0$, from (198). Furthermore, because $\widetilde{S}_2(\bar{p})$ is a strictly concave function of $\bar{p}$, from Lemma 5:

$$\frac{\partial \widetilde{S}_2(\bar{p})}{\partial \bar{p}} > 0 \;\text{ for all } \bar{p} < \bar{p}_{S_2 M}\,. \tag{259}$$

Observe that:

$$\bar{p}^* < \bar{p}_{V_2 m} < \bar{p}_{S_2 M}\,. \tag{260}$$

The first inequality in (260) reflects (258). The second inequality in (260) reflects Lemma 5. (236), (259), and (260) imply:

$$\frac{\partial \widetilde{S}_2(\bar{p}^*)}{\partial \bar{p}} > 0\,. \tag{261}$$

$\frac{\partial \widetilde{S}_2(\bar{p}^*)}{\partial \bar{p}} > 0$ (from (261)), $\frac{\partial \widetilde{V}_2(\bar{p}^*)}{\partial \bar{p}} < 0$ (by assumption), and $\bar{p}^* \in (\bar{p}_1, \bar{p}_2)$ (by assumption) imply that consumer surplus increases and R's revenue declines as $\bar{p}$ increases above $\bar{p}^*$. Therefore, $\bar{p}^*$ cannot be the welfare-maximizing value of $\bar{p}$. Hence, by contradiction, $\frac{\partial V_2(\bar{p}^*)}{\partial \bar{p}} \geq 0$. Consequently, (257) implies that $\frac{\partial \bar{p}^*}{\partial d} \leq 0$. ■

Lemma 6. *When* $\bar{p} \in (\bar{p}_1, \bar{p}_2)$*:*

(i) $\frac{dq_A}{dc} < 0\,,\ \frac{dq_N}{dc} > 0\,,\ \frac{dQ^R}{dc} \gtreqless 0 \Leftrightarrow k_A \gtreqless b\,,$

$\frac{dq}{dc} < 0$ *if* b *is sufficiently small, and* $\frac{dQ}{dc} < 0\,;$

(ii) $\frac{dq_A}{dc_A} < 0\,,\ \frac{dq_N}{dc_A} > 0\,,\ \frac{dQ^R}{dc_A} < 0\,,\ \frac{dq}{dc_A} > 0\,,$ *and* $\frac{dQ}{dc_A} < 0\,;$

(iii) $\frac{dq_A}{dc_N} > 0\,,\ \frac{dq_N}{dc_N} < 0\,,\ \frac{dQ^R}{dc_N} \lesseqgtr 0 \ \Leftrightarrow\ k_A \gtreqless b\,,$

$\frac{dq}{dc_N} \gtreqless 0 \ \Leftrightarrow\ k_A \gtreqless b\,,$ *and* $\frac{dQ}{dc_N} \lesseqgtr 0 \ \Leftrightarrow\ k_A \gtreqless b\,.$

Proof. (20), (21), (24), and (2) imply:

$$\frac{dq_A}{dc} = -\frac{b\left[\,b + k^R\,\right]}{D}, \quad \frac{dq_N}{dc} = \frac{b\left[\,k_A + k^R\,\right]}{D}, \quad \text{and}$$
$$\frac{dQ}{dc} = -\frac{k^R\left[\,k_A + k_N\,\right] + k_A\left[\,b + k_N\,\right]}{D}\,. \tag{262}$$

(262) implies that because $D > 0$:

$$\frac{dq_A}{dc} < 0\,, \quad \frac{dq_N}{dc} > 0\,, \quad \text{and} \quad \frac{dQ}{dc} < 0\,. \tag{263}$$

(262) also implies that because $D > 0$:

$$\frac{dQ^R}{dc} \stackrel{s}{=} b\left[\,k_A + k^R\,\right] - b\left[\,b + k^R\,\right] = b\left[\,k_A - b\,\right] \gtreqless 0 \quad \Leftrightarrow \quad k_A \gtreqless b\,.$$

(23) implies that because $D > 0$:

$$\frac{dq}{dc} \stackrel{s}{=} -\left[\,k_N\left(k_A + k^R\right) + k_A\,k^R + 2\,b\,k_A - b^2\,\right] < 0 \ \text{ if } b \text{ is sufficiently small.}$$

(2), (20), (21), (23), and (24) imply:

$$\frac{dq_A}{dc_A} = -\frac{1}{D}\left[\,3\,b^2 + 2\,b\left(k + k_N + k^R\right) + k\left(k_N + k^R\right)\,\right], \quad \frac{dq}{dc_A} = \frac{b\left[\,b + k_N\,\right]}{D}$$
$$\frac{dq_N}{dc_A} = \frac{b\left[\,b + 2\,k^R\,\right] + k\left[\,b + k^R\,\right]}{D}, \quad \text{and} \quad \frac{dQ}{dc_A} = -\frac{\left[\,b + k\,\right]\left[\,b + k_N\,\right]}{D}\,. \tag{264}$$

(264) implies that because $D > 0$:

$$\frac{dq_A}{dc_A} < 0\,, \quad \frac{dq_N}{dc_A} > 0\,, \quad \frac{dq}{dc_A} > 0\,, \text{ and } \frac{dQ}{dc_A} < 0\,. \tag{265}$$

(264) also implies that because $D > 0$:

$$\frac{dQ^R}{dc_A} \stackrel{s}{=} b\left[\,b + 2\,k^R\,\right] + k\left[\,b + k^R\,\right] - 3\,b^2 - 2\,b\left[\,k + k_N + k^R\,\right] - k\left[\,k_N + k^R\,\right]$$

$$= \; -2\,b^2 + k\left[\,b + k^R\,\right] - 2\,b\,[\,k + k_N\,] - k\left[\,k_N + k^R\,\right]$$

$$= \; -2\,b^2 - b\,k \; - 2\,b\,k_N \; - k\,k_N \; < \; 0\,.$$

(2), (20), (21), (23), and (24) imply that because $D > 0$:

$$\frac{dq_A}{dc_N} \; = \; \frac{[\,2\,b + k\,]\left[\,b + k^R\,\right]}{D} \; > \; 0, \quad \frac{dq}{dc_N} \; = \; \frac{b\,[\,k_A - b\,]}{D} \; \gtreqless \; 0 \; \Leftrightarrow \; k_A \; \gtreqless \; b$$

$$\frac{dq_N}{dc_N} \; = \; -\frac{[\,2\,b + k\,]\left[\,k_A + k^R\,\right]}{D} \; < \; 0\,, \; \text{ and}$$

$$\frac{dQ}{dc_N} \; = \; \frac{[\,b + k\,]\,[\,k_A - b\,]}{D} \; \gtreqless \; 0 \; \Leftrightarrow \; k_A \; \gtreqless \; b\,. \tag{266}$$

(266) implies that because $D > 0$:

$$\frac{dQ^R}{dc_A} \; \stackrel{s}{=} \; [\,2\,b + k\,]\left[\,b + k^R\,\right] - [\,2\,b + k\,]\left[\,k_A + k^R\,\right] \; \stackrel{s}{=} \; b - k_A\,. \; \blacksquare$$

Proposition 8. *When* $\bar{p}^* \in (\bar{p}_1, \bar{p}_2)$*: (i)* $\frac{d\,\overline{p}^*}{dc_A} \; > \; 0$*; (ii)* $\frac{d\,\overline{p}^*}{dk_A} > 0$*; (iii)* $\frac{d\,\overline{p}^*}{dc} \; > \; 0$*; and (iv)* $\frac{d\,\overline{p}^*}{dc_N} < 0\,.$

Proof. (173) implies that consumer surplus is:

$$S \; = \; a\,Q - \frac{1}{2}\,b\,Q^2 - p\,[\,q + q_N\,] - \overline{p}\;q_A$$

$$= \; a\,Q - \frac{1}{2}\,b\,Q^2 - p\,[\,q + q_N + q_A\,] + [\,p - \overline{p}\,]\,q_A$$

$$= \; a\,Q - \frac{1}{2}\,b\,Q^2 - [\,a - b\,Q\,]\,Q + [\,p - \overline{p}\,]\,q_A$$

$$= \; \frac{1}{2}\,b\,Q^2 + [\,p - \overline{p}\,]\,q_A \; = \; \frac{1}{2}\,b\,Q^2 + [\,a - b\,Q - \overline{p}\,]\,q_A$$

$$= \; \frac{b}{2}\,Q^2 + [\,a - \overline{p}\,]\,q_A - b\,Q\,q_A\,. \tag{267}$$

(267) implies that $\overline{p}^*$ is the solution to:

$$\underset{\overline{p}}{\text{Maximize}} \;\; W \; = \; \frac{b}{2}\,Q^2 + [\,a - \overline{p}\,]\,q_A - b\,Q\,q_A - \; d\,\overline{p}\;q_A - d\,a\,q_N + d\,b\,Q\,q_N\,. \tag{268}$$

(192) and (268) imply that for $\bar{p} \in [\,\bar{p}_1, \bar{p}_2\,]$:

$$\frac{dW}{d\overline{p}} \;=\; b\,Q\left[\frac{[\,b+k\,]\,[\,b+k_N\,]}{D}\right] + [\,a-\overline{p}\,]\left[\frac{3\,b^2+2\,b\left[\,k+k_N+k^R\,\right]+k\left[\,k_N+k^R\,\right]}{D}\right]$$

$$-\;\; q_A - b\,Q\left[\frac{3\,b^2+2\,b\left[\,k+k_N+k^R\,\right]+k\left[\,k_N+k^R\,\right]}{D}\right] - b\,q_A\left[\frac{[\,b+k\,]\,[\,b+k_N\,]}{D}\right]$$

$$-\; d\,q_A - d\,\overline{p}\left[\frac{3\,b^2+2\,b\left[\,k+k_N+k^R\,\right]+k\left[\,k_N+k^R\,\right]}{D}\right]$$

$$-\; d\,a\left[-\,\frac{b\left[\,b+2\,k^R\,\right]+k\left[\,b+k^R\,\right]}{D}\right]$$

$$+\;\; d\,b\,Q\left[-\,\frac{b\left[\,b+2\,k^R\,\right]+k\left[\,b+k^R\,\right]}{D}\right] + d\,b\,q_N\left[\frac{[\,b+k\,]\,[\,b+k_N\,]}{D}\right] \;=\; 0$$

$$\Leftrightarrow\;\; b\,[\,b+k\,]\,[\,b+k_N\,]\,Q + [\,a-\overline{p}\,]\left\{\,3\,b^2+2\,b\left[\,k+k_N+k^R\,\right]+k\left[\,k_N+k^R\,\right]\,\right\}$$

$$-\;\; D\,q_A - b\left\{\,3\,b^2+2\,b\left[\,k+k_N+k^R\,\right]+k\left[\,k_N+k^R\,\right]\,\right\}Q - b\,[\,b+k\,]\,[\,b+k_N\,]\,q_A$$

$$-\; d\,D\,q_A - d\,\overline{p}\left\{\,3\,b^2+2\,b\left[\,k+k_N+k^R\,\right]+k\left[\,k_N+k^R\,\right]\,\right\}$$

$$+\; d\,a\;\left\{\,b\left[\,b+2\,k^R\,\right]+k\left[\,b+k^R\,\right]\,\right\}$$

$$-\; d\,b\left\{\,b\left[\,b+2\,k^R\,\right]+k\left[\,b+k^R\,\right]\,\right\}Q + d\,b\,[\,b+k\,]\,[\,b+k_N\,]\,q_N \;=\; 0$$

$$\Leftrightarrow\;\; \{\,b\,[\,b+k\,]\,[\,b+k_N\,] - b\left[\,3\,b^2+2\,b\left(k+k_N+k^R\right)+k\left(k_N+k^R\right)\,\right]$$

$$-\; d\,b\left[\,b\left(b+2\,k^R\right)+k\left(b+k^R\right)\,\right]\,\}\,Q$$

$$-\;\{\,D+b\,[\,b+k\,]\,[\,b+k_N\,]+d\,D\,\}\;q_A + d\,b\,[\,b+k\,]\,[\,b+k_N\,]\;q_N$$

$$-\;\{\,3\,b^2+2\,b\left[\,k+k_N+k^R\,\right]+k\left[\,k_N+k^R\,\right]$$

$$+\;\; d\left[\,3\,b^2+2\,b\left(k+k_N+k^R\right)+k\;\left(k_N+k^R\right)\,\right]\,\}\,\overline{p}$$

$$+\;\{\,3\,b^2+2\,b\left[\,k+k_N+k^R\,\right]+k\left[\,k_N+k^R\,\right]$$

$$+\;\; d\;\left[\,b\left(b+2\,k^R\right)+k\left(b+k^R\right)\,\right]\,\}\,a \;=\; 0\,. \tag{269}$$

The coefficient on Q in (269) is:

$b\,[\,b+k\,]\,[\,b+k_N\,] - b\left[\,3\,b^2+2\,b\left(k+k_N+k^R\right)+k\left(k_N+k^R\right)\,\right]$

$$- \, d\, b \left[b \left(b + 2\, k^R \right) + k \left(b + k^R \right) \right]$$

$$= \; b \, [\, b^2 + b\, k_N + b\, k + k\, k_N - 3\, b^2 - 2\, b\, k - 2\, b\, k_N - 2\, b\, k^R - k\, k_N - k\, k^R$$
$$- \, d\, b \left(b + 2\, k^R \right) - dk \left(b + k^R \right)]$$

$$= \; b \left[- 2\, b^2 - b\, k_N - b\, k - 2\, b\, k^R - k\, k^R - b^2 d - 2\, b\, d\, k^R - b\, d\, k - d\, k\, k^R \right]$$

$$= \; - b \, [\, 2\, b^2 + b\, k + b\, k_N + 2\, b\, k^R + k\, k^R + b^2 d$$
$$+ \; 2\, b\, d\, k^R + b\, d\, k + d\, k\, k^R \,] \; < \; 0 \, . \tag{270}$$

(2) implies that the coefficient on $- q_A$ in (269) is:

$$[\, 1 + d \,]\, D + b\, [\, b + k \,]\, [\, b + k_N \,]$$
$$= \; [\, 1 + d \,] \left\{ [\, 2\, b + k \,] \left[k_N \left(k_A + k^R \right) + k_A\, k^R \right] + b\, k_A\, [\, 3\, b + 2\, k \,] - b^2\, [\, b + k \,] \right\}$$
$$+ \, b\, [\, b + k \,]\, [\, b + k_N \,] \; > \; 0 \;\; \text{because} \;\; D \; > \; 0 \, . \tag{271}$$

(269) – (271) imply that if $\bar{p}^* \in (\bar{p}_1, \bar{p}_2)$, $\bar{p}^*$ is determined by:

$$G - g\; \bar{p}^* \; = \; 0, \;\; \text{where} \tag{272}$$

$$G \; \equiv \; d\, b\, [\, b + k \,]\, [\, b + k_N \,]\; q_N$$
$$+ \; \{\, 3\, b^2 + 2\, b \left[k + k_N + k^R \right] + k \left[k_N + k^R \right]$$
$$+ \;\; d \; \left[b \left(b + 2\, k^R \right) + k \left(b + k^R \right) \right] \} \, a$$
$$- \; b\, [\, 2\, b^2 + b\, k + b\, k_N + 2\, b\, k^R + k\, k^R + b^2 d$$
$$+ \;\; 2\, b\, d\, k^R + b\, d\, k + d\, k\, k^R \,]\, Q$$
$$- \;\; \{\, [\, 1 + d \,]\, D + b\, [\, b + k \,]\, [\, b + k_N \,] \,\}\; q_A \, , \;\; \text{and}$$

$$g \; \equiv \; \{\, 3\, b^2 + 2\, b \left[k + k_N + k^R \right] + k \left[k_N + k^R \right]$$
$$+ \;\; d \left[3\, b^2 + 2\, b \left(k + k_N + k^R \right) + k \; \left(k_N + k^R \right) \right] \} \; > \; 0 \, . \tag{273}$$

To prove that $\frac{d\,\overline{p}^*}{dc_A} \; > \; 0$, observe from (273) that $\frac{dg}{d\,\overline{p}} \; = \; 0$. Therefore, (272) implies that for parameter x:

$$[\, G_x - \; \overline{p}^* \; g_x \,]\, dx + [\, G_{\overline{p}} - \; g \,]\, d\,\overline{p}^* \; = \; 0 \quad \Rightarrow \quad \frac{d\,\overline{p}^*}{dx} \; = \; \frac{G_x - \; \overline{p}\; g_x}{g - G_{\overline{p}}} \, . \tag{274}$$

(2) and (273) imply that because $D > 0$:

$$G_{c_A} \; = \; d\, b\, [\, b + k \,]\, [\, b + k_N \,] \; \frac{dq_N}{dc_A}$$

$$- b\,[\,2\,b^2 + b\,k + b\,k_N + 2\,b\,k^R + k\,k^R + b^2 d$$

$$+ \;\; 2\,b\,d\,k^R + b\,d\,k + d\,k\,k^R\,]\,\frac{dQ}{dc_A}$$

$$- \;\; \{\,[\,1 + d\,]\,D + b\,[\,b + k\,]\,[\,b + k_N\,]\,\}\,\frac{dq_A}{dc_A}$$

$$> \; 0 \;\; \text{if} \;\; \frac{dq_A}{dc_A} \; < \; 0\,, \;\; \frac{dq_N}{dc_A} \; > \; 0\,, \;\; \frac{dQ}{dc_A} \; < \; 0\,. \tag{275}$$

(265) and (275) imply that because $D > 0$:

$$G_{c_A} \; > \; 0\,. \tag{276}$$

(2) and (273) imply that because $D > 0$:

$$G_{\overline{p}} \;\; = \;\; d\,b\,[\,b + k\,]\,[\,b + k_N\,]\,\frac{dq_N}{d\overline{p}}$$

$$- \; b\,[\,2\,b^2 + b\,k + b\,k_N + 2\,b\,k^R + k\,k^R + b^2 d$$

$$+ \;\; 2\,b\,d\,k^R + b\,d\,k + d\,k\,k^R\,]\,\frac{dQ}{d\overline{p}}$$

$$- \;\; \{\,[\,1 + d\,]\,D + b\,[\,b + k\,]\,[\,b + k_N\,]\,\}\,\frac{dq_A}{d\overline{p}}$$

$$< \; 0 \;\; \text{if} \;\; \frac{dq_A}{d\overline{p}} \; > \; 0\,, \;\; \frac{dq_N}{d\overline{p}} \; < \; 0\,, \text{and} \;\; \frac{dQ}{d\overline{p}} \; > \; 0\,. \tag{277}$$

(192) implies that because $D > 0$:

$$\frac{dq_A}{d\overline{p}} \; > \; 0\,, \;\; \frac{dq_N}{d\overline{p}} \; < \; 0\,, \;\; \text{and} \;\; \frac{dQ}{d\overline{p}} \; > \; 0\,. \tag{278}$$

(277) and (278) imply that because $D > 0$:

$$G_{\overline{p}} \; < \; 0\,. \tag{279}$$

(273) implies:

$$g_{c_A} \;\; = \;\; 0\,. \tag{280}$$

(273), (274), (276), (279), and (280) imply that because $D > 0$:

$$\frac{d\,\overline{p}^*}{dc_A} \;\; = \;\; \frac{G_{c_A}}{g - G_{\overline{p}}} \;\; > \;\; 0\,.$$

To prove that $\frac{d\overline{p}^*}{dc} \; > \; 0$, observe that (2) and (273) imply that because $D > 0$:

$$G_c \;\; = \;\; d\,b\,[\,b + k\,]\,[\,b + k_N\,]\,\frac{dq_N}{dc}$$

$$- \, b\,[\,2\,b^2 + b\,k + b\,k_N + 2\,b\,k^R + k\,k^R + b^2 d$$

$$+ \; 2\,b\,d\,k^R + b\,d\,k + d\,k\,k^R \,]\,\frac{dQ}{dc}$$

$$- \; \{\,[\,1+d\,]\,D + b\,[\,b+k\,]\,[\,b+k_N\,]\,\}\,\frac{dq_A}{dc}$$

$$> \; 0 \;\text{ if }\; \frac{dq_A}{dc} \; < \; 0\,, \;\; \frac{dq_N}{dc} \; > \; 0\,, \text{ and } \; \frac{dQ}{dc} \; < \; 0\,. \tag{281}$$

(263) and (281) imply that because $D > 0$:

$$G_c \; > \; 0\,. \tag{282}$$

(273) implies:

$$g_c \; = \; 0\,. \tag{283}$$

(273), (274), (279), (282), and (283) imply that because $D > 0$:

$$\frac{d\,\overline{p}^*}{dc} \; = \; \frac{G_c}{g - G_{\overline{p}}} \; > \; 0\,.$$

To prove that $\frac{\partial \bar{p}^*}{\partial c_N} \; < \; 0$, observe that (195) implies that for $\bar{p} \in (\bar{p}_1, \bar{p}_2)$:

$$\frac{\partial V_2(\,\bar{p}\,)}{\partial \bar{p}} \; = \; q_A + \bar{p}\,\frac{\partial q_A}{\partial \bar{p}} + P(Q)\,\frac{\partial q_N}{\partial \bar{p}} - b\,q_N\,\frac{\partial Q}{\partial \bar{p}} \tag{284}$$

$$\Rightarrow \;\; \frac{\partial^2 V_2(\,\bar{p}\,)}{\partial \bar{p}\,\partial c_N} \; = \; \frac{\partial q_A}{\partial c_N} + \bar{p}\,\frac{\partial^2 q_A}{\partial \bar{p}\,\partial c_N} + P\,(Q)\,\frac{\partial^2 q_N}{\partial \bar{p}\,\partial c_N}$$

$$- \; b\,\frac{\partial Q}{\partial c_N}\,\frac{\partial q_N}{\partial \bar{p}} - b\,q_N\,\frac{\partial^2 Q}{\partial \bar{p}\,\partial c_N} - b\,\frac{\partial q_N}{\partial c_N}\,\frac{\partial Q}{\partial \bar{p}}$$

$$= \; \frac{\partial q_A}{\partial c_N} - b\,\frac{\partial Q}{\partial c_N}\,\frac{\partial q_N}{\partial \bar{p}} - b\,\frac{\partial q_N}{\partial c_N}\,\frac{\partial Q}{\partial \bar{p}}\,. \tag{285}$$

The last equality in (285) holds because $\frac{\partial^2 q_A}{\partial \bar{p}\,\partial c_N} \; = \; \frac{\partial^2 q_N}{\partial \bar{p}\,\partial c_N} \; = \; \frac{\partial^2 Q}{\partial \bar{p}\,\partial c_N} \; = \; 0$ when $\bar{p} \in (\bar{p}_1, \bar{p}_2)$, from Lemma A2.

(2) and Lemma A2 imply that when $\bar{p} \in (\bar{p}_1, \bar{p}_2)$:

$$\frac{\partial Q}{\partial c_N}\,\frac{\partial q_N}{\partial \bar{p}} + \frac{\partial q_N}{\partial c_N}\,\frac{\partial Q}{\partial \bar{p}}$$

$$\stackrel{s}{=} \; [\,b+k\,]\,[\,k_A - b\,]\,\big[\,b\,\big(b + 2\,k^R\big) + k\,\big(b + k^R\big)\,\big] - [\,2\,b+k\,]\,\big[\,k_A + k^R\,\big]\,[\,b+k\,]\,[\,b+k_N\,]$$

$$\stackrel{s}{=} \; [\,k_A - b\,]\,\big[\,b\,\big(b + 2\,k^R\big) + k\,\big(b + k^R\big)\,\big] - [\,2\,b+k\,]\,\big[\,k_A + k^R\,\big]\,[\,b+k_N\,]$$

$$= \; k_A\,\big[\,b\,\big(b + 2\,k^R\big) + k\,\big(b + k^R\big)\,\big] - b\,\big[\,b\,\big(b + 2\,k^R\big) + k\,\big(b + k^R\big)\,\big]$$

$$- \; k_A\,[\,2\,b+k\,]\,[\,b+k_N\,] - k^R\,[\,b+k_N\,]\,[\,2\,b+k\,]$$

$$= \; k_A\,\big[\,b\,\big(b + 2\,k^R\big) + k\,\big(b + k^R\big) - (2\,b+k)\;(b+k_N)\,\big]$$

$$- \, b \left[\, b \left(b + 2 \, k^R \right) + k \left(b + k^R \right) \right] - k^R \left[\, b + k_N \, \right] \left[\, 2 \, b + k \, \right] . \tag{286}$$

The coefficient on k_A in (286) is:

$$\begin{aligned} & b \left[\, b + 2 \, k^R \, \right] + k \left[\, b + k^R \, \right] - \left[\, 2 \, b + k \, \right] \left[\, b + k_N \, \right] \\ & \quad = \; b^2 + 2 \, b \, k^R + b \, k + k \, k^R - 2 \, b^2 - 2 \, b \, k_N - k \, b - k \, k_N \\ & \quad = \; 2 \, b \, k^R + k \, k^R - b^2 - 2 \, b \, k_N - k \, k_N \, . \end{aligned} \tag{287}$$

(286) and (287) imply that because $k_A \leq k_N$:

$$\begin{aligned} & \frac{\partial Q}{\partial c_N} \, \frac{\partial q_N}{\partial \bar{p}} + \frac{\partial q_N}{\partial c_N} \, \frac{\partial Q}{\partial \bar{p}} \\ & \stackrel{s}{=} \; k_A \left[\, 2 \, b \, k^R + k \, k^R - b^2 - 2 \, b \, k_N - k \, k_N \, \right] - b \left[\, b \left(b + 2 \, k^R \right) + k \left(b + k^R \right) \right] \\ & \qquad \qquad - \, k^R \left[\, b + k_N \, \right] \left[\, 2 \, b + k \, \right] \\ & \leq \; k_A \left[\, 2 \, b \, k^R + k \, k^R - b^2 - 2 \, b \, k_N - k \, k_N \, \right] - b \left[\, b \left(b + 2 \, k^R \right) + k \left(b + k^R \right) \right] \\ & \qquad \qquad - \, k^R \left[\, b + k_A \, \right] \left[\, 2 \, b + k \, \right] \\ & = \; k_A \left[\, 2 \, b \, k^R + k \, k^R - b^2 - 2 \, b \, k_N - k \, k_N - k^R \left(2 \, b + k \right) \right] \\ & \qquad - \, b \left[\, b \left(b + 2 \, k^R \right) + k \left(b + k^R \right) \right] - k^R \, b \, \left[\, 2 \, b + k \, \right] \\ & = \; k_A \left[- b^2 - 2 \, b \, k_N - k k_N \, \right] - b \left[\, b \left(b + 2 \, k^R \right) + k \left(b + k^R \right) \right] - k^R \, b \left[\, 2 \, b + k \, \right] \; < \; 0 \, . \end{aligned} \tag{288}$$

Because $\frac{\partial q_A}{\partial c_N} \; > \; 0$ when $\bar{p} \in (\bar{p}_1, \bar{p}_2)$, from Lemma A2, (285) and (288) imply:

$$\frac{\partial^2 V_2(\bar{p})}{\partial \bar{p} \, \partial c_N} \; = \; \frac{\partial q_A}{\partial c_N} - b \left[\, \frac{\partial Q}{\partial c_N} \, \frac{\partial q_N}{\partial \bar{p}} + \frac{\partial q_N}{\partial c_N} \, \frac{\partial Q}{\partial \bar{p}} \, \right] \; > \; 0 \, . \tag{289}$$

(234) and (284) imply:

$$\begin{aligned} \frac{\partial S_2(\bar{p})}{\partial \bar{p}} \; & = \; a \, \frac{\partial Q}{\partial \bar{p}} - b \, Q \, \frac{\partial Q}{\partial \bar{p}} - q_A - \bar{p} \, \frac{\partial q_A}{\partial \bar{p}} - P\left(Q\right) \left[\, \frac{\partial q_N}{\partial \bar{p}} + \frac{\partial q}{\partial \bar{p}} \, \right] + b \, \frac{\partial Q}{\partial \bar{p}} \, \left[\, q_N + q \, \right] \\ & = \; a \, \frac{\partial Q}{\partial \bar{p}} - b \, Q \, \frac{\partial Q}{\partial \bar{p}} - P\left(Q\right) \, \frac{\partial q}{\partial \bar{p}} + b \, \frac{\partial Q}{\partial \bar{p}} \, q \\ & \qquad \qquad - \, q_A - \bar{p} \, \frac{\partial q_A}{\partial \bar{p}} - P\left(Q\right) \, \frac{\partial q_N}{\partial \bar{p}} + b \, \frac{\partial Q}{\partial \bar{p}} \, q_N \\ & = \; a \, \frac{\partial Q}{\partial \bar{p}} - b \, Q \, \frac{\partial Q}{\partial \bar{p}} - P\left(Q\right) \, \frac{\partial q}{\partial \bar{p}} + b \, \frac{\partial Q}{\partial \bar{p}} \, q - \frac{\partial V_2(\bar{p})}{\partial \bar{p}} \end{aligned}$$

$$\Rightarrow \quad \frac{\partial^2 S_2(\bar{p})}{\partial \bar{p}\, \partial c_N} = a \, \frac{\partial^2 Q}{\partial \bar{p}\, \partial c_N} - b \, Q \, \frac{\partial^2 Q}{\partial \bar{p} \partial c_N} - b \, \frac{\partial Q}{\partial \bar{p}} \, \frac{\partial Q}{\partial c_N} - \frac{\partial^2 V_2(\bar{p})}{\partial \bar{p}\, \partial c_N}$$

$$- \; P(Q) \, \frac{\partial^2 q}{\partial \bar{p}\, \partial c_N} + b \, \frac{\partial Q}{\partial c_N} \, \frac{\partial q}{\partial \bar{p}} + b \, \frac{\partial^2 Q}{\partial \bar{p} \partial c_N} \, q + b \, \frac{\partial Q}{\partial \bar{p}} \, \frac{\partial q}{\partial c_N} \, . \tag{290}$$

Lemma A2 implies that $\frac{\partial^2 Q}{\partial \bar{p}\, \partial c_N} = \frac{\partial^2 q}{\partial \bar{p} \partial c_N} = 0$ when $\bar{p} \in (\bar{p}_1, \bar{p}_2)$. Therefore, because $Q = Q^R + q$, (290) implies:

$$\frac{\partial^2 S_2(\bar{p})}{\partial \bar{p}\, \partial c_N} = -\, b \, \frac{\partial Q}{\partial \bar{p}} \, \frac{\partial Q}{\partial c_N} - \frac{\partial^2 V_2(\bar{p})}{\partial \bar{p}\, \partial c_N} + b \, \frac{\partial Q}{\partial c_N} \, \frac{\partial q}{\partial \bar{p}} + b \, \frac{\partial Q}{\partial \bar{p}} \, \frac{\partial q}{\partial c_N}$$

$$= -\, b \, \frac{\partial Q}{\partial \bar{p}} \, \frac{\partial Q^R}{\partial c_N} - \frac{\partial^2 V_2(\bar{p})}{\partial \bar{p}\, \partial c_N} + b \, \frac{\partial Q}{\partial c_N} \, \frac{\partial q}{\partial \bar{p}} \, . \tag{291}$$

(285) and (291) imply that because $Q^R = q_A + q_N$:

$$\frac{\partial^2 S_2(\bar{p})}{\partial \bar{p}\, \partial c_N} = -\, b \, \frac{\partial Q}{\partial \bar{p}} \, \frac{\partial Q^R}{\partial c_N} - \left[\frac{\partial q_A}{\partial c_N} - b \, \frac{\partial Q}{\partial c_N} \, \frac{\partial q_N}{\partial \bar{p}} - b \, \frac{\partial q_N}{\partial c_N} \, \frac{\partial Q}{\partial \bar{p}} \right] + b \, \frac{\partial Q}{\partial c_N} \, \frac{\partial q}{\partial \bar{p}}$$

$$= -\, b \, \frac{\partial Q}{\partial \bar{p}} \left[\frac{\partial Q^R}{\partial c_N} - \frac{\partial q_N}{\partial c_N} \right] - \frac{\partial q_A}{\partial c_N} + b \, \frac{\partial Q}{\partial c_N} \left[\frac{\partial q}{\partial \bar{p}} + \frac{\partial q_N}{\partial \bar{p}} \right]$$

$$= -\, b \, \frac{\partial Q}{\partial \bar{p}} \, \frac{\partial q_A}{\partial c_N} - \frac{\partial q_A}{\partial c_N} + b \, \frac{\partial Q}{\partial c_N} \left[\frac{\partial q}{\partial \bar{p}} + \frac{\partial q_N}{\partial \bar{p}} \right] . \tag{292}$$

(2), (292), and Lemma A2 imply:

$$\frac{\partial^2 S_2(\bar{p})}{\partial \bar{p}\, \partial c_N} \stackrel{s}{=} -\, b \, [\, b + k \,] \, [\, b + k_N \,] \, [\, 2\, b + k \,] \left[\, b + k^R \, \right] - \, [\, 2\, b + k \,] \left[\, b + k^R \, \right] D$$

$$+ \;\; b \, [\, -\, (b + k) \, (k_A - b) \,] \left[-b \, (b + k_N) - b \left(b + 2\, k^R \right) - k \left(b + k^R \right) \right]$$

$$= -\, b \, [\, b + k \,] \, [\, b + k_N \,] \, [\, 2\, b + k \,] \left[\, b + k^R \, \right] - \, [\, 2\, b + k \,] \left[\, b + k^R \, \right] D$$

$$+ \;\; b \, [\, b + k \,] \, [\, k_A - b \,] \left[\, b \, (b + k_N) + b \left(b + 2\, k^R \right) + k \left(b + k^R \right) \right] . \tag{293}$$

(2) and (293) imply:

$$\frac{\partial^2 S_2(\bar{p})}{\partial \bar{p}\, \partial c_N} = -\, b \, [\, b + k \,] \, [\, b + k_N \,] \, [\, 2\, b + k \,] \left[\, b + k^R \, \right]$$

$$+ \, b \, [\, b + k \,] \, [\, k_A - b \,] \left[\, b \, (b + k_N) + b \left(b + 2\, k^R \right) + k \left(b + k^R \right) \right]$$

$$- \, [\, 2\, b + k \,] \left[\, b + k^R \, \right]$$

$$\cdot \, \left[\, (2\, b + k) \left(k_N \left[\, k_A + k^R \, \right] + k_A k^R \right) + b \, k_A \, (3\, b + 2\, k) - b^2 \, (b + k) \, \right]$$

$$
\begin{aligned}
&= b\,[\,b+k\,]\,[\,k_A-b\,]\left[\,b\,(b+k_N)+b\left(b+2\,k^R\right)+k\left(b+k^R\right)\right] \\
&\quad -[\,2\,b+k\,]\left[\,b+k^R\,\right] \\
&\quad \cdot\left[\,b\,(b+k)\,(b+k_N)+(2\,b+k)\left(k_N\left[\,k_A+k^R\,\right]+k_A\,k^R\right)+b\,k_A\,(3\,b+2\,k)-b^2\,(b+k)\right] \\
&= b\,[\,b+k\,]\,[\,k_A-b\,]\left[\,b\,(b+k_N)+b\left(b+2\,k^R\right)+k\left(b+k^R\right)\right] \\
&\quad -[\,2\,b+k\,]\left[\,b+k^R\,\right]\left[\,b\,(b+k)\;k_N+(2\,b+k)\left(k_N\left[\,k_A+k^R\,\right]+k_A k^R\right)+b\,k_A\,(3\,b+2\,k)\right] \\
&= -b\,[\,b+k\,]\,b\left[\,b\,(b+k_N)\;+b\left(b+2\,k^R\right)+k\left(b+k^R\right)\right] \\
&\quad +b\,[\,b+k\,]\,k_A\left[\,b\,(b+k_N)+b\left(b+2\,k^R\right)+k\left(b+k^R\right)\right] \\
&\quad -[\,2\,b+k\,]\left[\,b+k^R\,\right]\left[\,b\,(b+k)\,k_N+(2\,b+k)\left(k_N\left[\,k_A+k^R\,\right]+k_A k^R\right)\right] \\
&\quad -k_A\,[\,2\,b+k\,]\left[\,b+k^R\,\right]\;b\;[\,3\,b+2\,k\,] \\
&= -b\,[\,b+k\,]\,b\left[\,b\,(\,b+k_N)+b\left(b+2\,k^R\right)+k\left(b+k^R\right)\right] \\
&\quad -[\,2\,b+k\,]\left[\,b+k^R\,\right]\left[\,b\,(b+k)\;k_N+(2\,b+k)\;k_N\,k^R\,\right] \\
&\quad +k_A\left\{\,b\,[\,b+k\,]\left[\,b\,(b+k_N)+b\left(b+2\,k^R\right)+k\left(b+k^R\right)\right]\right. \\
&\qquad\qquad \left.-[\,2\,b+k\,]\left[\,b+k^R\,\right]\left[(\,2\,b+k)\;\left(k_N+k^R\right)+b\,(3\,b+2\,k)\right]\right\}.
\end{aligned}
\tag{294}
$$

The coefficient on k_A in (294) is:

$$
\begin{aligned}
&b\,[\,b+k\,]\left[\,b\,(b+k_N)\;+b\left(b+2\,k^R\right)+k\left(b+k^R\right)\right] \\
&\qquad -[\,2\,b+k\,]\left[\,b+k^R\,\right]\left[\,(2\,b+k)\;\left(k_N+k^R\right)\;+b\,(3\,b+2\,k)\right] \\
&= k_N\left[\,b\,(b+k)\;b-(2\,b+k)\;\left(b+k^R\right)\;(2\,b+k)\right] \\
&\qquad +b\,[\,b+k\,]\left[\,b^2\;+b\left(b+2\,k^R\right)+k\left(b+k^R\right)\right] \\
&\qquad -[\,2\,b+k\,]\left[\,b+k^R\,\right]\left[\,(2\,b+k)\;k^R\;+b\,(3\,b+2\,k)\right] \\
&= k_N\left[\,b\,(b+k)\;b-(2\,b+k)\;\left(b+k^R\right)\;(2\,b+k)\right] \\
&\qquad +k^R\left[\,b\,(b+k)\;(2\,b+k)-(2\,b+k)\;\left(b+k^R\right)\;(2\,b+k)\right] \\
&\qquad +b\,[\,b+k\,]\left[\,2\,b^2+k\,b\,\right]-[\,2\,b+k\,]\left[\,b+k^R\,\right]b\,[\,3\,b+2\,k\,].
\end{aligned}
\tag{295}
$$

The coefficient on k_N in (295) is:

$$
b\,[\,b+k\,]\,b-[\,2\,b+k\,]\left[\,b+k^R\,\right][\,2\,b+k\,]\;<\;0\,. \tag{296}
$$

The inequality in (296) holds because $b < b + k^R$, $b + k < 2\,b + k$, and $b < 2\,b + k$.

The coefficient on k^R in (295) is:

$$b\,[\,b+k\,]\,[\,2\,b+k\,] - [\,2\,b+k\,]\left[\,b+k^R\,\right][\,2\,b+k\,] \;<\; 0\,. \tag{297}$$

The inequality in (297) holds because $b < b + k^R$ and $b + k < 2\,b + k$.

The last line in (295) is:

$$\begin{aligned} & b\,[\,b+k\,]\left[\,2\,b^2+k\,b\,\right] - [\,2\,b+k\,]\left[\,b+k^R\,\right] b\,[\,3\,b+2\,k\,] \\ & = \; b^2\,[\,b+k\,]\,[\,2\,b+k\,] - [\,2\,b+k\,]\left[\,b+k^R\,\right] b\,[\,3\,b+2\,k\,] \\ & \overset{s}{=} \; b\,[\,b+k\,] - \left[\,b+k^R\,\right][\,3\,b+2\,k\,] \;<\; 0\,. \end{aligned} \tag{298}$$

The inequality in (298) holds because $b < b + k^R$ and $b + k < 3\,b + 2\,k$.

(295) – (298) imply that the coefficient on k_A in (294) is negative. Therefore, (294) implies:

$$\frac{\partial^2 S_2(\bar{p}\,)}{\partial \bar{p}\,\partial c_N} \;<\; 0\,. \tag{299}$$

$\bar{p}^*$ satisfies:

$$\frac{\partial S_2(\bar{p}^*)}{\partial \bar{p}} - d\,\frac{\partial V_2(\bar{p}^*)}{\partial \bar{p}} \;=\; 0\,. \tag{300}$$

Totally differentiating (300) with respect to c_N provides:

$$\frac{\partial^2 S_2(\bar{p}^*)}{(\partial \bar{p}\,)^2}\,\frac{\partial \bar{p}^*}{\partial c_N} + \frac{\partial^2 S_2(\bar{p}^*)}{\partial \bar{p}\,\partial c_N} - d\left[\frac{\partial^2 V_2(\bar{p}^*)}{(\partial \bar{p})^2}\,\frac{\partial \bar{p}^*}{\partial c_N} + \frac{\partial^2 V_2(\bar{p}^*)}{\partial \bar{p}\,\partial c_N}\right] \;=\; 0$$

$$\Rightarrow \quad \frac{\partial \bar{p}^*}{\partial c_N} \;=\; -\,\frac{\frac{\partial^2 S_2(\bar{p}^*)}{\partial \bar{p}\,\partial c_N} - d\,\frac{\partial^2 V_2(\bar{p}^*)}{\partial \bar{p}\,\partial c_N}}{\frac{\partial^2 S_2(\bar{p}^*)}{(\partial \bar{p})^2} - d\,\frac{\partial^2 V_2(\bar{p}^*)}{(\partial \bar{p})^2}} \;=\; -\,\frac{\frac{\partial^2 S_2(\bar{p}^*)}{\partial \bar{p}\,\partial c_N} - d\,\frac{\partial^2 V_2(\bar{p}^*)}{\partial \bar{p}\,\partial c_N}}{\frac{\partial^2 W_2(\bar{p}^*)}{(\partial \bar{p})^2}} \;<\; 0\,.$$

The inequality follows from (289) and (299), because $\frac{\partial^2 W_2(\bar{p}^*)}{(\partial \bar{p})^2} \;<\; 0$ (from (172) and Lemmas 4 and 5).

To prove that $\frac{d\bar{p}^*}{dk_A} > 0$, observe that (2) implies:

$$\frac{\partial D}{\partial k_A} \;=\; [\,2\,b+k\,]\left[\,k_N+k^R\,\right] + b\,[\,3\,b+2\,k\,] \;>\; 0\,. \tag{301}$$

(20) and (301) imply that for $\bar{p} \in (\bar{p}_1, \bar{p}_2)$:

$$\frac{dq_A}{dk_A} \;=\; -\,\frac{q_A}{D}\,\frac{\partial D}{\partial k_A} \;<\; 0\,. \tag{302}$$

(29) and (302) imply that for $\bar{p} \in (\bar{p}_1, \bar{p}_2)$:

$$\frac{\partial q_N}{\partial k_A} = - \left[\frac{k_A + k^R}{b + k^R} \right] \frac{\partial q_A}{\partial k_A} - \left[\frac{1}{b + k^R} \right] q_A$$

$$= \left[\frac{k_A + k^R}{b + k^R} \right] \frac{q_A}{D} \frac{\partial D}{\partial k_A} - \left[\frac{1}{b + k^R} \right] q_A \stackrel{s}{=} \left[k_A + k^R \right] \frac{1}{D} \frac{\partial D}{\partial k_A} - 1 \,. \tag{303}$$

(2), (301), and (303) imply that $\frac{\partial q_N}{\partial k_A} > 0$ because:

$$\frac{\partial q_N}{\partial k_A} > 0 \;\Leftrightarrow\; \left[k_A + k^R \right] \frac{\partial D}{\partial k_A} > D$$

$$\Leftrightarrow\; [\, 2\, b + k \,] \left[k_N \left(k_A + k^R \right) + k_A\, k^R \right] + b\, k_A [\, 3\, b + 2\, k \,] - b^2 [\, b + k \,]$$

$$< \left[k_A + k^R \right] \left[(2\, b + k) \left(k_N + k^R \right) + b\, (3\, b + 2\, k) \right]$$

$$\Leftrightarrow\; [\, 2\, b + k \,] \left[k_N \left(k_A + k^R \right) + k_A\, k^R \right] + b\, k_A [\, 3\, b + 2\, k \,] - b^2 [\, b + k \,]$$

$$< [\, 2\, b + k \,] \left[k_N + k^R \right] \left[k_A + k^R \right] + b \left[k_A + k^R \right] [\, 3\, b + 2\, k \,]$$

$$\Leftrightarrow\; [\, 2\, b + k \,] \left[k_N \left(k_A + k^R \right) + k_A\, k^R \right] + b\, k_A [\, 3\, b + 2\, k \,] - b^2 [\, b + k \,]$$

$$< [\, 2\, b + k \,] \left[k_N \left(k_A + k^R \right) + k^R\, k_A + \left(k^R \right)^2 \right] + b \left[k_A + k^R \right] [\, 3\, b + 2\, k \,]. \tag{304}$$

It is apparent that the inequality in (304) holds.

Because $Q(\bar{p})$ is linear in $\bar{p}$:

$$Q(\bar{p}) = Q(\bar{p}_1) + \frac{\partial Q}{\partial \bar{p}} [\, \bar{p} - \bar{p}_1 \,] \quad \text{for } \bar{p} \in (\bar{p}_1, \bar{p}_2) \,. \tag{305}$$

(24) implies that for $\bar{p} \in (\bar{p}_1, \bar{p}_2)$:

$$\frac{\partial Q}{\partial \bar{p}} = \frac{[\, b + k \,] \, [\, b + k_N \,]}{D} \,. \tag{306}$$

(301) and (306) imply:

$$\frac{\partial Q}{\partial \bar{p}\, \partial k_A} = - \frac{[\, b + k \,] [\, b + k_N \,]}{D^2} \frac{\partial D}{\partial k_A} < 0 \,. \tag{307}$$

(6) implies that $\bar{p}_1$ does not vary with k_A. Lemma A1 implies that $Q(\bar{p}_1)$ does not vary with k_A. Therefore, (305) and (307) imply that for $\bar{p} \in (\bar{p}_1, \bar{p}_2)$:

$$\frac{\partial Q(\bar{p})}{\partial k_A} = \frac{\partial Q}{\partial \bar{p}\, \partial k_A} [\, \bar{p} - \bar{p}_1 \,] < 0 \,. \tag{308}$$

In summary, (302), (304), and (308) imply:

$$\frac{dq_A}{dk_A} < 0\,, \quad \frac{dq_N}{dk_A} > 0\,, \quad \text{and} \quad \frac{dQ}{dk_A} < 0 \;\text{ for all } \bar{p} \in (\bar{p}_1, \bar{p}_2)\,. \tag{309}$$

(20) implies that for $\bar{p} \in (\bar{p}_1, \bar{p}_2)$:

$$\begin{aligned} D\, q_A \;=\; & \left[\, 3\, b^2 + 2\, b \left(k + k_N + k^R \right) + k\, \left(k_N + k^R \right) \right] [\, \bar{p} - c_A \,] \\ & + b \left[\, b + k^R \,\right] [\, a - c \,] - [\, 2\, b + k \,] \left[\, b + k^R \,\right] [\, a - c_N \,] \end{aligned}$$

which is not a function of k_A. Therefore, (273) implies:

$$\begin{aligned} G_{k_A} \;=\; & d\, b\, [\, b + k \,]\, [\, b + k_N \,]\, \frac{\partial q_N}{\partial k_A} \\ & -\; b \left[\, 2\, b^2 + b\, k + b\, k_N + 2\, b\, k^R + k\, k^R + b^2\, d + 2\, b\, d\, k^R + b\, d\, k + d\, k\, k^R \,\right] \frac{\partial Q}{\partial k_A} \\ & -\; b\, [\, b + k \,]\, [\, b + k_N \,]\, \frac{\partial q_A}{\partial k_A}\,. \end{aligned} \tag{310}$$

(309) and (310) imply:

$$G_{k_A} \;>\; 0\,. \tag{311}$$

(273) implies:

$$g_{k_A} \;=\; 0\,. \tag{312}$$

(273), (274), (279), (311), and (312) imply:

$$\frac{d\,\bar{p}^*}{dk_A} \;=\; \frac{G_{k_A}}{g - G_{\bar{p}}} \;>\; 0\,. \;\; \blacksquare$$

Finally, consider the benchmark setting in which the price of output supplied using A's input is set (exogenously) at $\bar{p}_A$ and the price of output supplied without using A's input is set (exogenously) at $\bar{p}_N$. In this setting, R chooses q_A and q_N to:

$$\text{Maximize} \;\; \bar{p}_A\, q_A + \bar{p}_N\, q_N - c_A\, q_A - \frac{k_A}{2}\, [\, q_A \,]^2 - c_N\, q_N - \frac{k_N}{2}\, [\, q_N \,]^2 - \frac{k^R}{2}\, [\, q_A + q_N \,]^2\,.$$

The necessary conditions for a solution to this problem, [P-E], are:

$$\begin{aligned} & \bar{p}_A - c_A - k_A\, q_A - k^R\, [\, q_A + q_N \,] \;\leq\; 0 \qquad q_A\, [\,\cdot\,] \;=\; 0\,; \\ & \bar{p}_N - c_N - k_N\, q_N - k^R\, [\, q_A + q_N \,] \;\leq\; 0 \qquad q_N\, [\,\cdot\,] \;=\; 0\,. \end{aligned} \tag{313}$$

(313) implies that if $q_A = 0$ and $q_N > 0$ at the solution to [P-E]:

$$\begin{aligned} & \bar{p}_N - c_N - k_N\, q_N - k^R\, q_N \;=\; 0 \;\;\Rightarrow\;\; q_N \;=\; \frac{\bar{p}_N - c_N}{k_N + k^R} \\ & \Rightarrow\;\; \frac{\partial q_N}{\partial \bar{p}_N} \;=\; \frac{1}{k_N + k^R} \;>\; 0 \quad \text{and} \quad \frac{\partial q_N}{\partial \bar{p}_A} \;=\; 0\,. \end{aligned}$$

(313) also implies that if $q_N = 0$ and $q_A > 0$ at the solution to [P-E]:

$$\bar{p}_A - c_A - k_A \, q_A - k^R \, q_A \;=\; 0 \;\Rightarrow\; q_A \;=\; \frac{\bar{p}_A - c_A}{k_A + k^R}$$

$$\Rightarrow\; \frac{\partial q_A}{\partial \bar{p}_A} \;=\; \frac{1}{k_A + k^R} \;>\; 0 \quad \text{and} \quad \frac{\partial q_A}{\partial \bar{p}_N} \;=\; 0\,.$$

(313) further implies that if $q_A > 0$ and $q_N > 0$ at the solution to [P-E]:

$$\bar{p}_A - c_A - k_A \, q_A \;=\; k^R \, [\, q_A + q_N \,] \;\Rightarrow\; q_A \left[\, k_A + k^R \,\right] \;=\; \bar{p}_A - c_A - k^R \, q_N$$

$$\Rightarrow\; q_A \;=\; \frac{\bar{p}_A - c_A - k^R \, q_N}{k_A + k^R}\,; \;\text{ and} \tag{314}$$

$$\bar{p}_N - c_N - k_N \, q_N \;=\; k^R \, [\, q_A + q_N \,] \;\Rightarrow\; q_N \left[\, k_N + k^R \,\right] \;=\; \bar{p}_N - c_N - k^R \, q_A$$

$$\Rightarrow\; q_N \;=\; \frac{\bar{p}_N - c_N - k^R \, q_A}{k_N + k^R}\,. \tag{315}$$

(314) and (315) imply:

$$q_N \;=\; \frac{\bar{p}_N - c_N}{k_N + k^R} - \frac{k^R}{k_N + k^R} \left[\frac{\bar{p}_A - c_A - k^R \, q_N}{k_A + k^R} \right]$$

$$\Rightarrow\; q_N \left[1 - \frac{\left(k^R\right)^2}{(k_A + k^R)\,(k_N + k^R)} \right] \;=\; \frac{\bar{p}_N - c_N}{k_N + k^R} - \frac{k^R \, [\, \bar{p}_A - c_A \,]}{[\, k_A + k^R \,]\,[\, k_N + k^R \,]}$$

$$\Rightarrow\; q_N \left[\left(k_A + k^R\right) \left(k_N + k^R\right) - \left(k^R\right)^2 \right] \;=\; [\, \bar{p}_N - c_N \,] \left[\, k_A + k^R \,\right] - k^R \, [\, \bar{p}_A - c_A \,]$$

$$\Rightarrow\; q_N \;=\; \frac{[\, \bar{p}_N - c_N \,] \left[\, k_A + k^R \,\right] - k^R \, [\, \bar{p}_A - c_A \,]}{[\, k_A + k^R \,]\,[\, k_N + k^R \,] - (k^R)^2}\,. \tag{316}$$

(314) and (316) imply:

$$q_A \;=\; \frac{\bar{p}_A - c_A}{k_A + k^R} - \frac{k^R}{k_A + k^R} \left\{ \frac{[\, \bar{p}_N - c_N \,] \left[\, k_A + k^R \,\right] - k^R \, [\, \bar{p}_A - c_A \,]}{[\, k_A + k^R \,]\,[\, k_N + k^R \,] - (k^R)^2} \right\}$$

$$= \; \frac{1}{[\, k_A + k^R \,] \left\{ [\, k_A + k^R \,]\,[\, k_N + k^R \,] - (k^R)^2 \right\}}$$

$$\cdot \left\{ [\, \bar{p}_A - c_A \,] \left\{ \left[\, k_A + k^R \,\right] \left[\, k_N + k^R \,\right] - \left(k^R\right)^2 \right\} - k^R \, [\, \bar{p}_N - c_N \,] \left[\, k_A + k^R \,\right] \right.$$

$$\left. + \left(k^R\right)^2 [\, \bar{p}_A - c_A \,] \right\}$$

$$= \frac{[\,\bar{p}_A - c_A\,]\left[\,k_A + k^R\,\right]\left[\,k_N + k^R\,\right] - k^R\,[\,\bar{p}_N - c_N\,]\left[\,k_A + k^R\,\right]}{[\,k_A + k^R\,]\left\{\,[\,k_A + k^R\,]\,[\,k_N + k^R\,] - (k^R)^2\,\right\}}$$

$$= \frac{[\,\bar{p}_A - c_A\,]\left[\,k_N + k^R\,\right] - k^R\,[\,\bar{p}_N - c_N\,]}{[\,k_A + k^R\,]\,[\,k_N + k^R\,] - (k^R)^2}\,. \tag{317}$$

(316) and (317) imply:

$$\frac{\partial q_A}{\partial \bar{p}_A} = \frac{k_N + k^R}{[\,k_A + k^R\,]\,[\,k_N + k^R\,] - (k^R)^2} > 0\,, \text{ and}$$

$$\frac{\partial\,(q_A + q_N)}{\partial \bar{p}_A} = \frac{k_N + k^R - k^R}{[\,k_A + k^R\,]\,[\,k_N + k^R\,] - (k^R)^2}$$

$$= \frac{k_N}{[\,k_A + k^R\,]\,[\,k_N + k^R\,] - (k^R)^2} > 0\,. \tag{318}$$

(318) implies that a reduction in $\bar{p}_A$ induces R to reduce both q_A and $q_A + q_N$.